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A Repeated Model of the International Monetary System without Direct Default Costs*

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Abstract

This paper considers a repeated version of the International Monetary System model of Farhi and Maggiori (2018) without a direct default cost. Issuance of a safe asset by the Hegemon is sustained by a no-default condition that trades off the short-term benefit of default against the continuation value of not defaulting. In this model, it is optimal for the Hegemon to maintain a constant issuance. The constant issuance policy may however, be unstable. In particular, the no-default condition links current issuance to issuance in the previous period. If the Hegemon adopts a simple, but short-sighted, heuristic rule that bases current issuance on the issuance in the previous period, then the constant issuance policy is unstable. If however, the Hegemon uses a heuristic that targets the demand for risky assets from the rest of the world, then the corresponding equilibrium is stable.

Keywords: International Monetary System; Reserve Currency; Safe Asset; Triffen Dilemma; Instability.

JEL CODES: C61; F33; G15.

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1 Introduction

Farhi and Maggiori (2018) provide a model of the international monetary system in which a Hegemon monopolistically issues a safe asset, denominated in its reserve currency, to the rest of the world. The key insight is that the international monetary system can be fragile and unstable. In particular, there is a Triffen dilemma: the issuance of the safe asset may collapse because of the temptation of the Hegemon to devalue its currency in a bad state. The main model presented in Farhi and Maggiori (2018) is a one-period model in which the temptation to devalue the currency is offset by a direct utility cost, or default cost, such as a trade sanction.

Farhi and Maggiori (2018) explain that the default cost might be interpreted as a reputation cost in a repeated model and present a formal analysis in their online appendix. The analysis is however, limited to a case where issuance is fixed over time and the issue of stability is not addressed. The purpose of this letter is to address these limitations. In particular, it shows that constancy of issuance can be optimal, but that if the Hegemon adopts a simple, heuristic policy rule, then the equilibrium outcome can also be unstable. In the concluding section, we suggest some further avenues for research.

2 One-Period Model

This section considers the basic one-period model. There is a single, non-storable (numeraire) consumption good. There is a risky asset in infinite supply. There are two states, high and low that occur with probability $(1-\lambda)$ and λ respectively. The expected gross return on the risky asset is $\bar{R} = (1-\lambda)R_{\mathrm{H}} + \lambda R_{\mathrm{L}}$ where R_{H} is the gross return in the high state and $R_{\mathrm{L}} < R_{\mathrm{H}}$ in the gross return in the low state. It is assumed that $\bar{R} > 1$. The variance of the return is $\sigma^2 = \lambda(1-\lambda)(R_{\mathrm{H}} - R_{\mathrm{L}})^2$. There are two agents: a Hegemon that issues a safe asset with gross return S in the reserve currency, and the rest of the world (RoW) that buys a portfolio of the safe and the risky assets from an initial endowment of w. The rest of the world has mean-variance preferences and seeks to maximize $\mathbb{E}[C] - (\gamma/2)\,\mathrm{var}[C]$, where C is consumption at the end of the period. The Hegemon is risk-neutral and can use the proceeds of the sale of the safe asset for consumption at the start of the period or to buy the risky asset and consume at the end of the period. It is assumed that the Hegemon's discount rate between the start and the end of the period is $\delta = 1/\bar{R}$.

With mean-variance preferences, the demand for the safe asset from the RoW is linear. Let $S(b)$ denote the gross return from the safe asset when the Hegemon issues an amount b. With the market for the safe asset in equilibrium, and linear demand, $S(b) = \bar{R}(1-\gamma\kappa(w-b))$, where $\kappa = \sigma^2/\bar{R}$ is the coefficient of dispersion (variance-to-mean ratio). To ensure that $S(0) \geq 0$, it is assumed $\gamma\kappa w \leq 1$. Let $P(b) := bS(b)$ be the amount the Hegemon pays out on the safe asset if it issues b. The function $P(b)$ is increasing and convex, with $P(0) = 0$ and $P'(b) = S(b) + bS'(b) = \bar{R}(1 - (\gamma\kappa(w - 2b))$. Since the Hegemon is risk-neutral and discounts at the rate $\delta = 1/\bar{R}$, the net benefit of issuance for the Hegemon is $f(b) := b\bar{R} - P(b)$, the difference between the expected return of investing b in the risky asset and the safe return it pays out. The per-period utility $f(b)$ is concave with $f(0) = f(w) = 0$. The Hegemon chooses the issuance b to maximize $f(b)$. The first-order condition, $P'(b) = \bar{R}$, satisfies the standard elasticity formula for a monopolist: $bS'(b) = \bar{R} - S(b)$. That is, the Hegemon recognizes that as issuance is increased, the nominal return on the safe asset increases, reducing profits. Therefore, the Hegemon restricts the issuance of the safe asset below the competitive level. In particular, with $P'(b) = \bar{R}(1 - \gamma\kappa(w - 2b))$, the Hegemon's optimal issuance is $b = w/2$ compared with the competitive equilibrium outcome of $b = w$. For future reference, let $b^* = w/2$ be the Hegemon's optimal issuance with corresponding utility $v^* = f(b^*) = \bar{R}\gamma\kappa(w/2)^2$.

In the model of Farhi and Maggiori (2018), the Hegemon can reduce its payment on the safe asset by devaluing its currency if the low state occurs. In particular, it can choose a currency value $e \in \{e_L, 1\}$ at the end of the period.[1] If the Hegemon devalues the currency to $e_L < 1$, the real return to the bond is $S(b)e_L$. The fiscal benefit for the Hegemon from a devaluation is $P(b)(1 - e_L)$. However, if the Hegemon devalues, it bears a direct default cost χ that is proportional to the devaluation. Therefore, the Hegemon devalues in the low state if and only if:

$$\underbrace{P(b)(1 - e_L)}_{\textit{Benefit of devaluation}} > \underbrace{\chi(1 - e_L)}_{\textit{Cost of devaluation}} .$$

That is, the Hegemon devalues if and only if $P(b) > \chi$.

3 Repeated Model

Consider an infinite horizon extension of the basic model with dates $t = 0, 1, 2, \ldots$, where there is no direct default cost: $\chi = 0$. The outcome of the risky asset is independently and identically distributed. The RoW is a sequence of one-period lived agents that invest when young (start of the period) and consume the returns when old (end of the period). At date t, the Hegemon issues b_t one-period bonds with a safe return $S(b_t)$ and can choose the exchange rate $e_t \in \{e_L, 1\}$ at the end of each period. The Hegemon discounts the payoffs at the end of the period as outlined in the baseline model.[2]

[1] For simplicity there is a binary choice between a devaluation to $e = e_L$ or no devaluation, $e = 1$. With $e_L = R_{\mathrm{L}}/R_{\mathrm{H}}$, if a devaluation is anticipated, the risky and safe assets are perfect substitutes.

[2] The Appendix considers a model where the Hegemon has quasi-hyperbolic preferences of the type introduced by Laibson (1997).

We look for an equilibrium with no devaluation. Suppose the Hegemon can choose a sequence of issuances $\{b_t\} := (b_t)_{t \in \mathbb{N}}$. Then

$$v_t = f(b_t) = b_t \bar{R} - P(b_t) \quad \text{and} \quad V_t := \sum_{\tau=0}^{\infty} \delta^\tau v_{t+\tau} = v_t + \delta V_{t+1}.$$

Let x_t denote the amount the Hegemon invests in the risky asset in period t. The Hegemon considers the decision on devaluation at the end of the period and only in the low state where it can choose $e_t \in \{e_L, 1\}$. If it chooses $e_t = 1$, then its end-of-period payoff is the return from the risky asset $x_t R_\text{L}$ less the amount paid out to the RoW on the safe bonds it issued at the start of the period, $P(b_t)$. The future payoff, at the end of the period, is V_{t+1}. If the Hegemon chooses $e_t = e_L$, then it receives its payoff $x_t R_\text{L}$ from the risky investment but the payout on the safe asset is reduced to $P(b_t)e_L$. However, if $e_t = e_L$, the risky asset and the safe asset are perfect substitutes and we assume that the demand for the safe asset will fall to zero in the future. Hence, the Hegemon will choose $e_t = 1$ when $x_t R_\text{L} - P(b_t) + \delta V_{t+1} \geq x_t R_\text{L} - P(b_t) e_L$. This can be equivalently written as:

$$\delta V_{t+1} \geq P(b_t)(1 - e_L). \tag{ND}$$

We will refer to this as the no-default condition. If the no-default condition holds when $b = b^* = w/2$, then the Hegemon will choose $b = b^*$ with a corresponding discounted utility $V^* = v^*/(1-\delta)$. With $\delta = 1/\bar{R}$, $V^* = \bar{R} v^* / (\bar{R} - 1)$.

4 Results

This section considers the implications of the no-default condition (ND) for the Hegemon's choice of $\{b_t\}$. Lemma 1 describes conditions on the parameter values such that $b_t = b^*$ for all t satisfies condition (ND). Lemma 2 discusses the properties of the difference equation $b_{t+1} - b_t$ that satisfies condition (ND). Proposition 1 summarizes the implications of these results.

Lemma 1. *The no-default constraint* (ND) *is satisfied for $b_t = b^*$ $\forall t \in \mathbb{N}$ when parameters w, γ, κ, e_L and $\bar{R}$ satisfy:*

$$w \geq \hat{w} := \left(\frac{2}{\gamma\kappa}\right)\left(1 - \frac{1}{e_L + \bar{R}^r(1 - e_L)}\right).$$

Proof. These are the parameter values such that $\delta V^* \geq P(b^*)(1 - e_L)$ when $b^* = w/2$ and $V^* = v^*/(1 - \delta)$, where $v^* = f(b^*) = \bar{R}\gamma\kappa(w/2)^2$, and $\delta = 1/\bar{R}$. ■

From now on, we assume that it is not possible to sustain the first-best issuance b^*; that is $w < \hat{w}$. When the condition (ND) holds with equality,

$$\delta V_{t+1} = P(b_t)(1 - e_L) = V_t - v_t = V_t - f(b_t) = V_t - \left(\bar{R}b_t - P(b_t)\right).$$

Hence, $V_t = \bar{R}b_t - e_L P(b_t)$. Since $P(b) = bS(b)$ and $S(b) = \bar{R}(1 - \gamma\kappa(w - b))$, this can be written as $V_t = F(b_t)$ where

$$F(b) := \bar{R}b - e_L P(b) = \bar{R}b\,(1 - e_L + e_L\gamma\kappa(w - b)).$$

The function $F(b)$ is concave with $F(0) = 0$. It is increasing in b for $b < \tilde{b}$ where $\tilde{b} > b^* = w/2$. Let $\tilde{V} := F(\tilde{b})$.[3] The inverse function $F^{-1}(V)$ is defined on $[0, \tilde{V}]$ and is increasing and convex with $F^{-1}(0) = 0$. Then, if condition (ND) holds with equality, and since $\delta = 1/\bar{R}$, it follows that $b_{t+1} = G(b_t)$ where:

$$G(b) := F^{-1}\left(\bar{R}P(b)(1 - e_L)\right).$$

Since both $P(b)$ and $F^{-1}(V)$ are increasing and convex, and $F^{-1}(0) = 0$ and $P(0) = 0$, it follows that $G(b)$ is increasing and convex and $G(0) = 0$. The corresponding difference equation is $D_G(b) = G(b) - b$ where $D_G\colon [0, \tilde{b}] \to \mathbb{R}$ is convex.[4] There are two roots $b_{[1,2]}$ of this difference equation, with $b_{[1]} = 0$. If $w \geq \hat{w}$, then by Lemma 1, there is a positive root $b_{[2]} \geq b^*$. If $w \leq \hat{w}/2$, then $b_{[2]} = 0$. The next lemma considers the case where $w \in (\hat{w}/2, \hat{w}]$.

Lemma 2. *If $w \in (\hat{w}/2, \hat{w}]$, then there is a unique positive root of the difference equation $D_G(b)$ where $b_{[2]} = b^\infty := w - \hat{w}/2 \in (0, b^*]$.*

Proof. Since $D_G(V)$ is concave, there are two roots. Since $G(0) = 0$, there is one root $b_{[1]} = 0$. It can be checked that with $b^\infty = w - \hat{w}/2$, $F(b^\infty) = \bar{R}P(b^\infty)(1 - e_L)$. Hence $b_{[2]} = w - \hat{w}/2$, which is increasing in w. If $w > \hat{w}/2$, then $b^\infty > 0$, and if $w = \hat{w}$, then $b^\infty = w/2 = b^*$. ∎

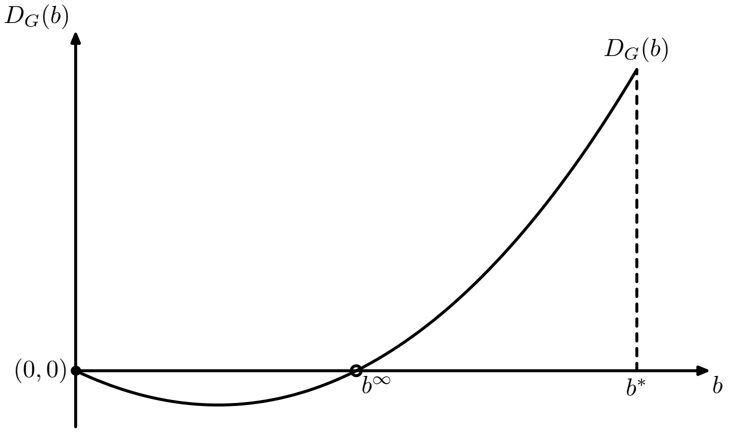
[3] It can be checked that:

$$\tilde{b} = \frac{1 - e_L}{2e_L\gamma\kappa} + \frac{w}{2} > \frac{w}{2} = b^*; \quad \tilde{V} = \frac{\bar{R}\,(1 - e_L(1 - \gamma\kappa w))^2}{4e_L\gamma\kappa}.$$

[4] It can be checked that:

$$D_G(b) = \frac{1 - e_L(1 - \gamma\kappa w) - \sqrt{(1 - e_L(1 - \gamma\kappa w))^2 - 4\bar{R}(1 - e_L)e_L\gamma\kappa(1 - \gamma\kappa(w - b))b}}{2e_L\gamma\kappa} - b.$$

There is an equivalent difference equation in V: $D_H(V) = H(V) - V$ where $H(V) := \bar{R}P(F^{-1}(V))(1 - e_L)$. The properties of $D_H(V)$ mirror those of $D_G(b)$.



Note: The case illustrated is for $w \in (\hat{w}/2, \hat{w})$. There are two roots: $b_{[1]} = 0$ and $b_{[2]} = b^\infty$ with $b^\infty \in (0, b^*)$. The root $b_{[1]}$ is stable, the root $b_{[2]}$ is unstable.

Figure 1: The Difference Equation $D_G(b) = F^{-1}(\bar{R}P(b)(1 - e_L)) - b$.

The difference equation $D_G(b)$ is illustrated in Figure 1 for a case where $w \in (\hat{w}/2, \hat{w})$. Since $D_G(b) < 0$ for $b \in (0, b^\infty)$ and $D_G(b) > 0$ for $b > b^\infty$, the root $b_{[2]} = b^\infty$ is unstable, whereas the root $b_{[1]} = 0$ is stable.

Corresponding to an issuance of b^∞, the per-period utility of the Hegemon is:

$$v^\infty = f(b^\infty) = \bar{R}b^\infty - P(b^\infty) = \bar{R}\left(w - \frac{\hat{w}}{2}\right)\left(\frac{\gamma\kappa\hat{w}}{2}\right).$$

With issuance constant at $b_t = b^\infty$, the Hegemon's discounted utility is:

$$V^\infty = \frac{\bar{R}}{\bar{R} - 1}v^\infty = F(b^\infty) = \bar{R}\left(w - \frac{\hat{w}}{2}\right)\left(1 - e_L + e_L\gamma\kappa\frac{\hat{w}}{2}\right).$$

Since the Hegemon's per-period utility $f(b)$ is increasing in b for $b < b^*$, it follows that when $w \in (\hat{w}/2, \hat{w})$, the Hegemon optimally chooses $b_t = b^\infty$ for all t. In their Appendix, Farhi and Maggiori (2018) assume that issuance is constant over time and show that the optimal constant issuance is $b = b^\infty$.[5] Thus, the result here shows that the assumption of constant issuance is without loss of generality. However, Lemma 2 also shows that the outcome $b_t = b^\infty$ for all t is fragile. Suppose that the Hegemon follows a simple heuristic rule, setting its issuance according

[5] In their Appendix, trigger strategies that depend on devaluation (but not issuance) are shown to support the issuance b^∞. In particular, there is a trigger strategy such that if the Hegemon devalues when $R < R_{\text{H}}$, then the rest of the world expects a devaluation in the low state in every subsequent period. Thus, in this continuation, the Hegemon sets $R = R_{\text{H}}$ and devalues if there is a low state, so that the safe and risky bonds are perfect substitutes. The Hegemon gets zero monopoly rents. On the other hand, when $R = R_{\text{H}}$ and the Hegemon devalues, there is no punishment. They further assume that the trigger strategy is played with a probability χ and the equilibrium in which the Hegemon devalues in the low state if $R = R_{\text{H}}$ and a safe outcome is possible is played with probability α. Here we are assuming $\chi = 1$ and $\alpha = 0$. Setting $\chi < 1$ or $\alpha > 0$ lowers the cost of default and would mean only lower issuance is sustainable.

to $b_{t+1} = G(b_t)$.[6] If $b_t = b^\infty$, then $b_{t+1} = b^\infty$ and issuance remains constant and optimal. However, consider, for example, an unexpected, one-period, negative shock to w, the income of RoW, at date t. Let $G_\varepsilon(b)$ denote the function $G(b)$ with the shock. With a negative shock, $G_\varepsilon(b) < G(b)$. Thus, if $b_t = b^\infty$ and $b_{t+1} = G_\varepsilon(b_t)$, then $b_{t+1} < b_t$, $b_{t+2} < G(b_{t+1})$ and so on. Issuance collapses unless the Hegemon re-optimizes.

The results in this section have shown that it is optimal for the Hegemon to choose a policy that involves keeping issuance constant. However, if the Hegemon adopts an intuitive heuristic to determine issuance recursively through the equation $b_{t+1} = G(b_t)$, then this constant issuance is unstable and may collapse after a one-period shock.

Proposition 1. *For $w \in (\hat{w}/2, \hat{w})$, a policy of constant issuance $b_t = b^\infty = w - \hat{w}/2$ is optimal. With a heuristic policy rule, $b_{t+1} = G(b_t)$, the policy of constant issuance is unstable.*

The instability result of Proposition 1 is different from the instability in the one-period model. In this repeated version, there is no default in equilibrium. In the one-period model of Farhi and Maggiori (2018), there may be default in equilibrium and multiple equilibria can occur: an equilibrium where the RoW investor correctly expects a devaluation and an equilibrium where the RoW investor correctly expects no devaluation. In the former case, the bond is risky and it is ex post optimal for the Hegemon to devalue in the low state. In the latter case, the bond is safe and, ex post, it is not optimal for the Hegemon to devalue in the low state. Farhi and Maggiori (2018) introduce a sunspot to coordinate on an equilibrium. In this case, the Hegemon faces a trade-off. It can keep issuance low or increase issuance with the risk that the sunspot might lead to the equilibrium with devaluation. It is this feature of the model that captures the *Triffin dilemma*. Issuing more nominal debt may lead to a situation where it will ultimately be optimal for the Hegemon to devalue its currency.

Here the instability result of Proposition 1 is based on a short-sighted heuristic where current issuance depends on the issuance in the previous period. Whilst this is a very natural heuristic, a differently targeted heuristic will have different stability properties. For example, if the Hegemon has a rule that targets the amount of the risky asset that the RoW buys, $z = w - b$, then the corresponding difference equation in z is stable at the root $z_{[2]} = w - b_{[2]}$. The choice of the rule matters for stability.

[6] This might be justified by supposing, for example, that the decisions of the Hegemon are made by successive short-sighed ruling parties.

5 Comparison with Risk-sharing Model

In this section, we examine a simple risk-sharing model with no default constraints to compare with the results of Section 4. The model is a simplified version of the *debt constrained* model of Kehoe and Levine (2001). There are two infinitely-lived and risk-averse agents with common utility function u and a common discount factor δ. There is one non-storable good and an aggregate endowment of $y = 2$. The endowment of each agent alternates between $y^{\mathrm{H}} = 1 + \epsilon$ and $y^{\mathrm{L}} = 1 - \epsilon$, where $\epsilon > 0$. In odd periods, agent 1 has the high endowment and agent 2 has the low endowment. In even periods, the situation is reversed. The agent with the high endowment can make a transfer to the agent with the low endowment. Let $\{p_t\}$ denote the sequence of these transfer payments. Assume that this transfer sequence must satisfy individual rationality or participation constraints that at each period the continuation payoff from this sequence of payments at least matches the discounted utility from endowment stream from that point on. The first-best outcome is to set $p_t = \epsilon$ at each period. Assume that parameters are such that this first-best outcome violates the participation constraints, and assume that there are positive transfers that satisfy the participation constraints.[7] In this case, (Kehoe and Levine, 2001, Propostion 3, p.582) show that there is a unique payment $p_{[2]} \in (0, \epsilon)$ that satisfies $p_{[2]} = J(p_{[2]})$ where:

$$J(p) := u^{-1}\left(\frac{u(1+\epsilon) - u(1+\epsilon-p)}{\delta} + u(1-\epsilon)\right) - (1-\epsilon).$$

The fixed point $p_{[2]}$ corresponds to the payment when the participation constraint for the agent with the high endowment binds. Equally, a solution with no transfer payment, $p_{[1]} = 0$, is also a fixed point of J. Consider the sequence $\{p_t\}$ determined by the recursion $p_{t+1} = J(p_t)$. This corresponds to a simple heuristic where the transfer payment is set so that the participation constraint of the agent with the high endowment, and making a transfer, binds in the expectation that the payment next period is set in the same way. That is, agents are short-sighted and only consider the current and the following period. Since $J(p)$ is increasing and convex with $J(0) = 0$, the root $p_{[2]}$ of this recursion is unstable.[8] Thus, we conclude that the instability

[7] The parameter restriction for this to be the case are:

$$\frac{u(1+\epsilon) - u(1)}{u(1) - u(1-\epsilon)} > \delta > \frac{1-\epsilon}{1+\epsilon}.$$

[8] Note that Kehoe and Levine (2001) consider a symmetric model with two identical agents. If say, one of the agents had more bargaining power at the initial period, then the transfer in the initial period might be different from $p_{[2]}$. For a case where the allocation in a transitory phase may be different from the long-run allocation, see, for example, Thomas and Worrall (1988) who consider a wage contracting model with one risk averse and one risk-neutral agent and solve for contracts on the constrained-efficient Pareto frontier.

result applies not just to the repeated Hegemon model, but more generally to some models of limited commitment such as Kehoe and Levine (2001).

6 Concluding Comments

This paper has presented a tractable repeated version of the model of Farhi and Maggiori (2018) in which optimal issuance of the safe asset by the Hegemon is constant. It has shown that this constant issuance can be fragile and unstable. Whilst the model considered here is the natural extension of Farhi and Maggiori (2018) to a repeated setting (it is the same as the model presented in their online appendix), it does have some limitations. For example, the Hegemon's choice of devaluation and issuance are taken sequentially rather than simultaneously and the choice of devaluation is binary rather than continuous. The RoW has a one-period horizon. Nevertheless, the tracatability of the model suggests that these and other issues can be addressed in future research. One such extension is to multiple issuers of safe assets. With more than one issuer, the rents from issuance are diluted, but the choice of issuance and default will also depend on the choices of the other issuers. The tractability of the model studied here suggests that further research to address these issues is worthwhile.

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Appendix – Extension to Quasi-hyperbolic Discounting

In this appendix, we consider an extension where the Hegemon discounts the future using quasi-geometric discounting (see, for example, Laibson, 1997). There are two ways to introduce quasi-geometric discounting into the model. One method is to introduce an extra discounting between the periods and another is to introduce an extra discounting within the period. Here we concentrate on the first alternative.[9] Let μ denote an extra impatience parameter between the end of the current period and the start of the next period. The case considered in the main text has $\mu = 1$. In this appendix, we consider $\mu < 1$. In this case, the Hegemon's continuation utility discounted to the start of the period is:

$$(\delta)v_t + (\mu\delta)(\delta v_{t+1}) + (\mu\delta^2)(\delta v_{t+2}) + \cdots = \delta X_t,$$

where $X_t = (1-\mu)v_t + \mu V_t$ and $V_t = \sum_{\tau=0}^{\infty} \delta^\tau v_{t+\tau}$. Since the discount factor between the start and the end of the period is unchanged, the Hegemon's consumption choice is unchanged from the case where $\mu = 1$. In particular, if $\delta = 1/\bar{R}$, the Hegemon's choice of investment in the risky asset at date t is indeterminate, $x_t \in [0, b]$, and the Hegemon's per-period payoff, discounted back to the start of the period, is $\delta b_t(\delta^{-1} - S(b_t)) = \delta b_t(\bar{R} - S(b_t))$ with the first-order condition $b_t S'(b_t) = \bar{R} - S(b_t)$. The no-default condition is written as:

$$\underbrace{x_t R_L - b_t S(b_t) + \mu\delta X_{t+1}}_{\text{expected discounted payoff at the end of period } t \text{ if no default}} \geq \underbrace{x_t R_L - e_L b_t S(b_t)}_{\text{expected discounted payoff at the end of period } t \text{ if default}}.$$

Since $X_t = (1-\mu)v_{t+1} + \mu V_{t+1}$, the no-default condition can be written as:

$$\mu\delta X_{t+1} = \mu\delta\left((1-\mu)v_{t+1} + \mu V_{t+1}\right) \geq P(b_t)(1-e_L). \tag{NDA}$$

If $\mu = 1$, the no-default constraint in (NDA) reduces to $\delta V_{t+1} \geq P(b_t)(1-e_L)$, which is condition (ND) in the main text.

The analysis can proceed as in the main text. First, assume that the no default constraint is binding. In this case, $P(b_t) = \mu\delta X_{t+1}/(1-e_L)$.[10] Write

$$v_t = K(X_{t+1}) := \bar{R}P^{-1}\left(\frac{\mu\delta X_{t+1}}{1-e_L}\right) - \frac{\mu\delta X_{t+1}}{1-e_L}.$$

[9] The results using the second alternative are available in the supplementary material.
[10] For simplicity we suppress the notational dependence of X and V on μ.

We have $K(0) = 0$ and $K'(0) = \mu/(\bar{R}A) > 0$ where $A := (1 - e_L)(1 - \gamma\kappa w)/(\gamma\kappa w)$. Since $X_t = (1 - \mu)v_t + \mu V_t$ and $V_t = v_t + \delta V_{t+1}$, we have a pair of difference equations:

$$X_{t+1} = K^{-1}\left(\frac{X_t - \mu V_t}{1 - \mu}\right), \tag{A.1}$$

$$V_{t+1} = \left(\frac{\bar{R}}{1 - \mu}\right)(V_t - X_t). \tag{A.2}$$

Using (A.1) and (A.2), the phase lines in (X, V)-space are:

$$V = L_V(X) := \left(\frac{\bar{R}}{\bar{R} - (1 - \mu)}\right) X, \tag{A.3}$$

$$V = L_X(X) := \frac{X}{\mu} - \left(\frac{1 - \mu}{\mu}\right) K(X). \tag{A.4}$$

The phase line $L_V(X)$ in (A.3), where V is constant, is a ray with slope $\bar{R}/(\bar{R} - (1 - \mu))$. Above the line, V is increasing, and below the line, V is decreasing. The phase line $L_X(X)$ in (A.4), where X is constant, also passes through the origin, and is convex. Its slope and curvature are given by:

$$L'_X(X) = \frac{1}{\mu} - \left(\frac{1 - \mu}{\mu}\right) K'(X) = \frac{1}{\mu}\left(1 + \frac{\mu(1 - \mu)}{\bar{R}(1 - e_L)}\right) - \left(\frac{1 - \mu}{1 - e_L}\right) P^{-1\prime}\left(\frac{\mu X}{\bar{R}(1 - e_L)}\right),$$

$$L''_X(X) = -\left(\frac{1 - \mu}{\mu}\right) K''(X) = -\frac{\mu(1 - \mu)}{\bar{R}(1 - e_L)^2} P^{-1\prime\prime}\left(\frac{\mu X}{\bar{R}(1 - e_L)}\right) > 0.$$

Below the line, X is increasing, and above the line, X is decreasing. Figure A.1 illustrates these phase lines for the case where $L_X(X)$ is increasing in X and indicates the dynamics in the phase space.

As can be seen from Figure A.1, equations (A.3) and (A.4) have a solution at $(0, 0)$ and at (V^∞, X^∞). Define

$$\hat{w}_\mu := \left(\frac{2}{\gamma\kappa}\right)\left(1 - \frac{\mu\left(1 - \frac{1-\mu}{\bar{R}}\right)}{e_L + \bar{R}(1 - e_L) - \left(1 - \mu\left(1 - \frac{1-\mu}{\bar{R}}\right)\right)}\right).$$

For $\mu = 1$, $\hat{w}_\mu = \hat{w}$ where $\hat{w}$ is defined in the main text. It can be checked that $\hat{w}_\mu$ is declining in μ, so that $\hat{w}_\mu > \hat{w}$ for $\mu < 1$. It can also be checked that the issuance corresponding to the point (V^∞, X^∞) is $b^\infty_\mu = w - \hat{w}_\mu/2$ where

$$X^\infty = \bar{R}^2(1 - e_L)\left(w - \frac{\hat{w}_\mu}{2}\right)\left(1 - \gamma\kappa\frac{\hat{w}_\mu}{2}\right), \quad \text{and} \quad V^\infty = \left(\frac{\bar{R}}{\bar{R} - (1 - \mu)}\right) X^\infty.$$

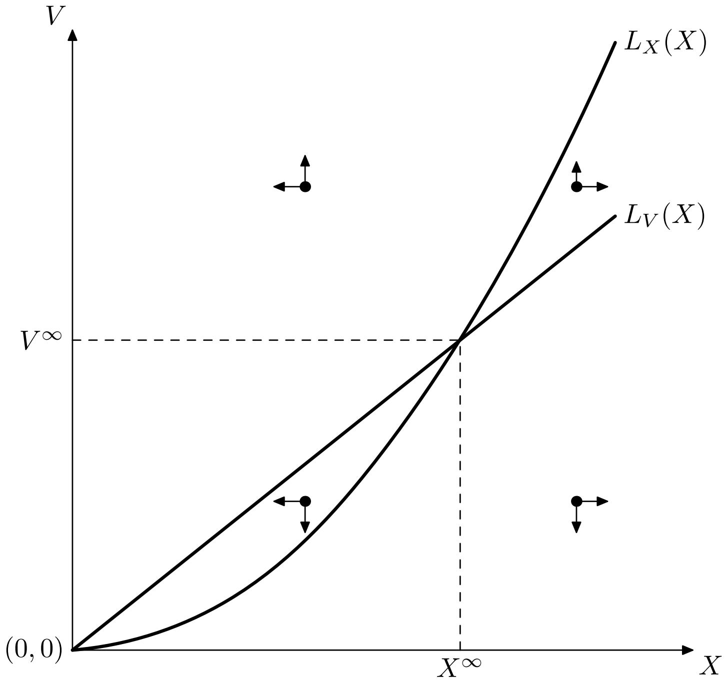

Figure A.1: Phase Diagram in (X, V)-space.

If $w > \hat{w}_\mu/2$, then $1 - \gamma\kappa\hat{w}_\mu/2 > 1 - \gamma\kappa w > 0$, and hence, $b^\infty_\mu > 0$, $X^\infty > 0$, and $V^\infty > 0$. Analogous to Lemma 1, the no-default constraint (NDA) is satisfied for $b_t = b^* = w/2\ \forall t \in \mathbb{N}$ when parameters $w, \gamma, \kappa, e_L, \mu$ and $\bar{R}$ satisfy $w \geq \hat{w}_\mu/2$. Analogous to Lemma 2, $b^\infty_\mu \in (0, b^*]$ when $w > \hat{w}_\mu/2$. Since the focus of the appendix is on the parameter μ, it is useful to express these conditions in terms of μ. It is also necessary to ensure that the phase line $L_X(X)$ is upward sloping for each μ. This is done in the next lemma.

Lemma A.1. *Assume $w \in (\hat{w}_\mu/2, \hat{w}_\mu]$. Furthermore, assume*

$$\mu \in \begin{cases} (\hat{\mu}, \check{\mu}_2) \bigcup (\check{\mu}_1, 1], & \text{if } \bar{R} \leq (4A)^{-1} \\ (\hat{\mu}, 1] & \text{otherwise}. \end{cases} \tag{A.5}$$

where $\check{\mu}_1 = (1 + \sqrt{1 - 4\bar{R}A})/2$, $\check{\mu}_2 = (1 - \sqrt{1 - 4\bar{R}A})/2$, and $\hat{w}_{\hat{\mu}} = 2w$. Then, $L_X(X)$ is increasing in X, and $b^\infty_\mu \in (0, b^*]$.

Proof. Since $L_X(X)$ is convex, it is increasing if $L'_X(0) > 0$. Also, $X^\infty > 0$ when $L'_X(0) < L'_V(0) = \bar{R}/(\bar{R} - (1 - \mu))$. That is, we need to check if

$$L'_X(0) = \frac{1}{\mu} - \left(\frac{1-\mu}{\mu}\right) K'(0) > 0 \quad \text{and} \quad L'_X(0) = \frac{1}{\mu} - \left(\frac{1-\mu}{\mu}\right) K'(0) < \frac{\bar{R}}{\bar{R} - (1-\mu)}.$$

Since $K'(0) = \mu/(\bar{R}A)$, these inequalities can be written as:

$$\mu^2 - \mu + \bar{R}A > 0, \tag{A.6}$$

$$\mu^2 + (\bar{R} - 1)\mu - (\bar{R} - 1)\bar{R}A > 0. \tag{A.7}$$

The first inequality (A.6) has a minimum at $\mu = 1/2$, when $\bar{R} = (4A)^{-1}$, and hence, it is always satisfied if

$$\bar{R}A > \frac{1}{4}.$$

If this inequality is reversed, $\bar{R} \le (4A)^{-1}$, then there is a $\breve{\mu}_1 \in (1/2, 1)$ and a $\breve{\mu}_2 \in (0, 1/2)$ given by

$$\breve{\mu}_1 = \frac{1}{2}\left(1 + \sqrt{1 - 4\bar{R}A}\right) \quad \text{and} \quad \breve{\mu}_2 = \frac{1}{2}\left(1 - \sqrt{1 - 4\bar{R}A}\right)$$

such that (A.6) is satisfied for $\mu \ge \breve{\mu}_1$ and $\mu \le \breve{\mu}_2$. It can be seen that $\breve{\mu}_1$ is decreasing in $\bar{R}$ with $\breve{\mu}_1 < 1$ when $\bar{R} = 1$ and it attains its lowest value of one-half when $\bar{R} = (4A)^{-1}$. Likewise, $\breve{\mu}_2$ is increasing in $\bar{R}$ with $\breve{\mu}_2 > 0$ when $\bar{R} = 1$ and it attains its highest value of one-half when $\bar{R} = (4A)^{-1}$. In particular, $\breve{\mu}_1, \breve{\mu}_2 \to 1/2$ as $\bar{R} \to (4A)^{-1}$.

Now consider inequality (A.7). It is not satisfied at $\mu = 0$. However, solving $\hat{w}_{\hat{\mu}} = w$ gives:

$$\hat{\mu} = \frac{1}{2}\left(-(\bar{R} - 1) + \sqrt{(\bar{R} - 1)^2 + 4(\bar{R} - 1)\bar{R}A}\right). \tag{A.8}$$

Since $w > \hat{w}_\mu/2 > \hat{w}/2$, it can be checked that $\bar{R}A < 1 + A$. From (A.8), $\hat{\mu} = 0$ for $\bar{R} = 1$ and is increasing in $\bar{R}$ for $\bar{R} > 1$. In this case, inequality (A.7) is satisfied for $\mu > \hat{\mu}$. When $\bar{R} = (4A)^{-1}$, or equivalently, $4A\bar{R} = 1$, we have

$$\hat{\mu} = \frac{1}{2}\left(-(\bar{R} - 1) + \sqrt{(\bar{R} - 1)\bar{R}}\right).$$

If $\bar{R} = (4A)^{-1}$, then $A \le 1/4$, since $\bar{R} \ge 1$. We have that $\breve{\mu}_2 - \hat{\mu} = 0$ when $A = 0$ and this difference is increasing in A for $A \le 1/4$ for each $\bar{R} > 1$. In particular, the derivative of $\breve{\mu}_2 - \hat{\mu}$ with respect to A is:

$$\begin{aligned}\frac{\partial(\breve{\mu}_2 - \hat{\mu})}{\partial A} &= \bar{R}\left(\frac{1}{\sqrt{1-4A\bar{R}}} - \frac{\bar{R}-1}{\sqrt{(\bar{R}-1)(\bar{R} - (1-4A\bar{R}))}}\right) \\ &= \bar{R}\left(\frac{\sqrt{(\bar{R} - (1-4A\bar{R}))} - \sqrt{(\bar{R} - (1-4A\bar{R})) - 4A\left(\bar{R}\right)^2}}{\left(\sqrt{1-4A\bar{R}}\right)\left(\sqrt{(\bar{R} - (1-4A\bar{R}))}\right)}\right),\end{aligned}$$

which is positive for $4A\bar{R} \leq 1$. Thus, $\check{\mu}_2 > \hat{\mu}$ for $4A\bar{R} < 1$. We conclude that both (A.6) and (A.7) hold for $w > \hat{w}_\mu/2$ and for μ satisfying (A.5). ∎

It follows from the phase lines and phase diagram in Figure A.1 that under the conditions of Lemma A.1, the point (X^∞, V^∞) is unstable, where as $(0, 0)$ is stable. Thus, when there is quasi-geometric discounting with μ less than one and satisfying (A.5), then the results of Proposition 1 from the main text remain valid.

We can also consider the limit as $\mu \to 1$ to show that it converges to the case considered in the main text. Note that the inequality (A.6) is always satisfied (it is also satisfied for $\mu = 0$) in the limit. Inequality (A.7) is satisfied in the limit when

$$(\bar{R} - 1)A < 1 \quad \text{or, equivalently} \quad \left(w - \frac{\hat{w}}{2}\right)\left(\frac{e_L + \bar{R}(1 - e_L)}{w}\right) > 0,$$

which is the condition given in the main text. As $\mu \to 1$, $X_t \to V_t$ and the equations (A.1) and (A.2) reduce to

$$V_{t+1} = K^{-1}(v_t) \quad \text{and} \quad V_{t+1} = \bar{R}\left(V_t - v_t\right).$$

Hence,

$$V_{t+1} = K^{-1}(V_t - \delta V_{t+1}) \quad \text{or, equivalently} \quad V_{t+1} = \bar{R}\left(V_t - K(V_{t+1})\right).$$

Translating V_t in terms of b_t, this is the case studied in the main text.