SGPE Summer School 2016 Mathematics Exam

Question 1:

Let $f(x) = \ln(e^x + e^{-x})$.

(a)[6pt] Compute $f'(x)$ and $f''(x)$.

Answer: $f'(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$f''(x) = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

(b) [2pt] What is the minimum of f?

Answer: Set $f'(x) = 0$ to find minimum so $x = 0$

(c)[2pt] Is f convex, concave, or neither?

Answer: Since $f''(x) > 0$, it is convex

Question 2:

Evaluate the following limits:

(a)[2pt] $\lim\limits_{x\to 1} \dfrac{x^4 - 1}{x^2 - 1}$ (b)[2pt] $\lim\limits_{x\to p} \dfrac{x - p}{\ln x - \ln p}$ (c)[3pt] $\lim\limits_{x\to\infty} \frac{1}{x}\sin x$ (d)[3pt] $\lim\limits_{x\to -2} \dfrac{\frac{1}{x} + \frac{1}{2}}{x^3 + 8}$

Answer: (a) $\lim\limits_{x\to 1} \dfrac{(x^2 - 1)(x^2 + 1)}{x^2 - 1} = 2$

(b) Apply L'Hôpital rule and limit will be p.

(c) 0. Check the quiz 3 for an answer.

(d) $\lim\limits_{x\to -2} \dfrac{\frac{1}{x} + \frac{1}{2}}{x^3 + 8} = \lim\limits_{x\to -2} \dfrac{x + 2}{2x(x^3 + 8)} = \lim\limits_{x\to -2} \dfrac{(x + 2)}{2x(x + 2)(x^2 - 2x + 4)} = -\dfrac{1}{48}$

Question 3:

Determine if each of the following are functions. For those you consider function, determine domain and range; determine if they are continuous.

(a)[5pt] $y = \frac{1}{x}$ (b)[5pt] $y = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$ (c)[5pt] $y^2 = x$

Answer: (a) It is a function. Domain and range for this functions is all real numbers except zero. Continuous.

(b) Function. Domain is $\{-1, 1\}$ and range is $\{0\}$. Continuous.

(c) Not a function. To see take $x = 4$, then $y = 2$ and $y = -2$. Thus the other parts are not relevant.

Question 4:

Calculate the following

(a)[5pt] $\int_e^{e^3}(4xlnx)\mathrm{d}x$ (b)[5pt] $\lim_{x\to 0} x\ln x^2$ (c)[5pt]$\int_{-2}^{2}(6x^2+10x+3)\mathrm{d}x$

Answer: (a) Integration by parts, $5e^6 - e^2$ (b) L'Hôpital 0 (c) 44

Question 5:

A department store is fencing off part of the store for children to meet and be photographed with Santa Claus. They have decided to fence off a rectangular region of fixed area 800 ft^2. Fire regulations require that there be three gaps in the fencing: 6 ft openings on the two facing sides and a 10 ft opening on the remaining wall (the fourth side of the rectangle will be against the building wall). Find the dimensions that will minimize the length of fencing used. [10pt]

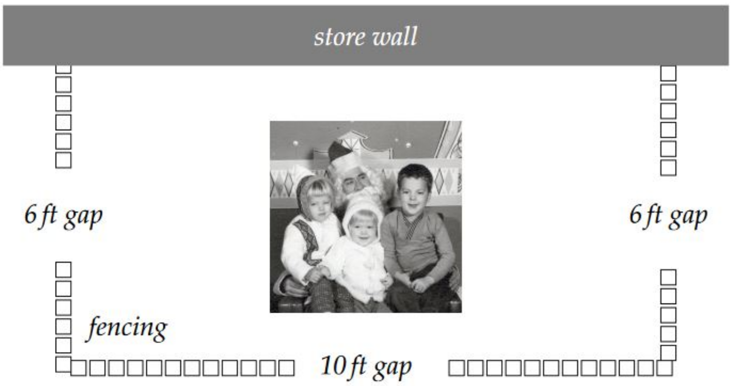

Figure 1: Picture of the store

Answer: See question 3 of Quiz 4.

Question 6:

Consider $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$. Write a general expression for k^{th} derivative $f^{(k)}$. [12pt]

Answer: $f^{(k)}(x) = (-1)^k k! x^{-(k+1)}$

Question 7:

Solve the following linear system of equations[12pt]

$$\begin{aligned} x + y + z &= 3 \\ y - z - 1 &= 0 \\ x + 3z - 2y &= 1 \end{aligned}$$

Answer: Substitution or elimination would work. $x = 4$, $y = 0$ and $z = -1$

Question 8:

Optimize the following function by (1) finding the critical value(s) at which the function is optimized and (2) testing the second-order condition to distinguish between a relative maximum or minimum and (3) the values of the relative extrema for the function.[16pt]

$$f(x) = x^2 + 6x - 96$$

Answer: $x = -3$ relative minimum and $f(-3) = -105$. For a detailed answer see quiz 4 question 1.