Simultaneous Nash Bargaining with Consistent Beliefs*

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Abstract

We propose and analyze a new solution concept, the R-solution, for three-person, transferable utility, cooperative games. In the spirit of the Nash Bargaining Solution, our concept is founded on the predicted outcomes of simultaneous, two-party negotiations that would be the alternative to the grand coalition. A crucial feature of the concept is that these possibly probabilistic predictions are based on consistent beliefs. We also propose a non-cooperative bargaining protocol that asymptotically implements the R-solution, as discounting vanishes. We analyze the properties of the R-solution and compare it with traditional solution concepts. The R-solution exists and is unique. It belongs to the bargaining set and to the core whenever the latter is not empty, and is their most egalitarian selection when the grand coalition can simply execute one of the three possible bilateral trades.

KEYWORDS: cooperative games, bargaining, endogenous disagreement payoffs, consistent beliefs, $R-$solution.

JEL classification numbers: C71, C78, L14.

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1 Introduction

We present a new solution concept for three-player, transferable utility, cooperative games, which can readily be applied to so-called multilateral bargaining problems, where not only the three-player coalition but also alternative bilateral coalitions can form. This solution concept can be viewed as a generalization of the Nash Bargaining Solution (NBS) based on consistent beliefs. In particular, as part of the solution concept we specify for each possible coalition both how players intend to share the surplus (if there is any) and the probability distribution over alternative events in case negotiations fail in the grand coalition. Thus, in contrast to alternative generalizations of the NBS, players' beliefs in each particular negotiation are mutually consistent (they are conditional distributions of the same joint probability distribution) and, moreover, satisfy a notion of consistency with respect to payoffs. We view this approach as the main contribution of the paper.

We also offer a non-cooperative bargaining protocol that asymptotically implements our solution concept (as the discount factor approaches one.) Such a non-cooperative approach allows us to illustrate the implications of our consistency requirements in terms of the flexibility of the bargaining protocol.

Our solution concept for cooperative games results from three basic postulates. First, following Harsanyi, the solution (agreement in the grand coalition) should "depend on the *pattern of cooperation and/or conflict* that would emerge among different groups of players should a general agreement fail." (Harsanyi, 1959, page 332.) In our case, the solution concept coincides with the generalized NBS of the grand coalition, where the disagreement points are endogenous and determined by the relative bargaining position of players in alternative, mutually exclusive bilateral coalitions.

Second, we model these bilateral negotiations as *simultaneous* Nash bargaining processes. Each player simultaneously bargains with the other two players, and the outcome of each negotiation can be represented by the two-player NBS. The disagreement points in each bilateral negotiation are endogenous and related to what each player can achieve in the alternative negotiations. In this respect we follow Bennett (1997), which is probably in spirit the paper closest to ours. Thus, our solution concept is motivated by economic problems where, in the absence of an agreement in the grand coalition, all three players are free to seek a bilateral agreement without facing any other constraint. We discuss various important economic examples below.

Third, and this is the novel postulate, disagreement points in any negotiation are based on consistent beliefs. In particular, we propose that the solution to the simultaneous bilateral negotiations must specify not only how the surplus would be shared by each coalition, but also the probability of success of that coalition. When two players engage in a bilateral negotiation, they should form beliefs about the likelihood that each of them would strike a deal with the third player, should their own negotiation fail. Given the initial probability distribution, consistency requires that these beliefs are obtained by Bayesian updating conditioning on the failure of that bilateral negotiation. In other words, all players share common priors that they use to compute conditional distributions by applying Bayes' rule. Moreover, this postulate also requires consistency between payoffs and beliefs. In particular, we assume that the probability of success of a bilateral coalition can only be significantly positive if both players expect to obtain in that negotiation a (weakly) higher payoff than in the their alternative negotiations. Finally, the probability distribution of the success of alternative bilateral coalitions, together with the prediction about how each coalition share their surplus, can be used to compute the disagreement points in the grand coalition.

Note that this third postulate implies that beliefs satisfy that: (a) from the point of view of each negotiating pair, the probabilities that the two players assign to the alternative negotiations succeeding should their negotiation fails, add up to one; and (b) the beliefs that any player holds in any negotiation are common, since they emanate from the same joint probability distribution conditional on the same event.

The main contribution of this paper is to demonstrate the implications of our belief consistency requirement on the solution of cooperative games. This can be illustrated by considering alternative solutions proposed in the literature. For instance, in Bennett (1997) the disagreement points in each bilateral negotiation are the payoffs that players obtain in their negotiation with the third player. This is equivalent to assuming that different players assign probability one to two different, mutually exclusive outcomes.[1] A similar problem arises when one considers the internal consistency properties that popular solution concepts have been shown to satisfy, and that are invoked to claim that these concepts are appropriate generalizations of the NBS for general games. More specifically, Hart and Mas-Colell (1989) characterize the Shapley value as the unique solution concept that satisfies one such definition of internal consistency. Using a different definition, Sobolev (1975) did the same for the nucleolus.[2] We will argue that both definitions involve specifications of reduced games that are consistent with the NBS only if payers hold inconsistent (incompatible) beliefs, just as in Bennett (1997).

As already mentioned, we postulate our solution concept also as a generalization of the NBS for three-player games. In general, one may extend

[1] The approach followed by Bennett (1997) does not always generate a unique prediction. In fact, a *solution* specifies the distribution of surplus in each alternative bilateral negotiation. The disagreement point in each negotiation is the payoff that each player would obtain in an alternative negotiation. Thus, the disagreement point in some negotiations may be outside the feasible set of that negotiation, which Bennett interprets as the failure of the negotiation. A predicted *outcome* specifies what negotiation will succeed.

[2] For the games we are considering, the nucleolus coincides with the kernel and also with the Nash set. See Peleg (1986) and Serrano end Shimomura (1998) for similar results for these two concepts for more general games.

the two-player NBS to more than two-player games following two alternative routes: backward and forward. The former involves a (perhaps axiomatic) characterization of a solution concept for n players, and by applying an internal consistency or reduced game property, finding that for $n = 2$ we obtain NBS. This is the route typically followed when authors prove internal consistency. The alternative forward path would take the NBS as the starting point and, based on a criterion for extrapolating the principles of that solution, attempt to define a solution for a larger class of games. Harsanyi (1959) offered an early example of an exercise of this kind, when he argued that the right generalization should propose disagreement points for the grand coalition, and these should come from mutually dependent payoffs predicted for smaller coalitions. For games where the alternative to the grand coalition are simultaneous bilateral negotiations, we follow this path by postulating that players should predict not only payoffs but also the probabilities of success of each agreement, so that payoffs *and* probabilities are consistent with each other. We think this approach is rooted in the main tradition of game theory.

In Section 2 we formalize these ideas. We show that our solution concept, the $R-$solution, always exists and is unique. As a step to defining the $R-$solution, we need to analyze what is known as the three player/three cake problem (Binmore, 1985). That is, a situation where only bilateral coalitions can form. We obtain a prediction for that case ($PSBN$) which is of independent interest for a variety of economic problems, like merger analysis.

Section 3 provides a dynamic, non-cooperative bargaining protocol whose stationary, subgame perfect equilibrium payoffs are arbitrarily close to those predicted by the $R-$solution as the discount factor approaches 1. Such a bargaining protocol offers further valuable interpretations of the requirements of the $R-$solution; in particular, concerning the consistency of beliefs.

Two particular features of the protocol are crucial to understand the predictions of the model. First, the game starts in a trilateral negotiation phase, but it may switch (with an exogenous and potentially very low probability) to a bilateral negotiation phase. Once the game enters a bilateral negotiation phase it never returns to a trilateral phase. Such asymmetric regime switching can be thought of as the formalization in a non-cooperative game of the first postulate discussed above, which grants the grand coalition a central role. Second, as far as bilateral negotiations are concerned, it separates the selection of the negotiation partners from the actual negotiation. This is achieved in a way that provides full flexibility in terms of which coalitions can be formed in a particular period and for any given choice of Nature. Thus, we avoid any artificial restrictions that distort the relative bargaining power evident in standard protocols.[3],[4]

In Section 4 we discuss the properties of the $R-$solution and its difference and similarity to alternative solution concepts. Contrary to the Shapley value, the $R-$solution is a selection of the core when the latter is not empty. When the core is empty, the Aumann-Maschler bargaining set (BS) is the most popular generalization. The BS contains the core and is never empty. We show that the $R-$solution is a selection of the BS. In fact, for superadditive, three-player, TU games, the BS (for the grand coalition) coincides with the core when the latter is not empty, and is a singleton when the core is empty. Thus, the $R-$solution coincides with the BS in the latter case (and hence with any selection of the BS). Moreover, if the grand coalition does not add any additional surplus, the $R-$solution is the most *egalitarian* se-

[3]In order to fix ideas in Section 3 we contrast our bargaining protocol with the one proposed by Fridolfsson and Stenneck (2005).

[4]Recently, Compte and Jehiel (2010) have defined the Coalitional NBS as the allocation of the core that maximizes the product of payoffs. Their main goal is to provide conditions under which the Coalitional NBS can be asymptotically implemented by a simple bargaining protocol that (contrary to Harsanyi's postulate) treats all coalitions, including the grand coalition, symmetrically. Their game cannot implement an efficient solution when the core is empty. When the core has a non-empty interior the Coalitional NBS and the $R-$solution select different allocations.

lection in the BS. Thus, it is more egalitarian than other, different selections of the core or the BS, like the nucleolus.[5]

We also discuss whether or not the $R-$solution satisfies several important axioms. We show that the $R-$solution satisfies symmetry, efficiency, and the dummy player axioms. Thus, it has to violate the additivity axiom since the Shapley value is the only solution concept that satisfies all four. Indeed, the $R-$solution is not additive. However, we argue that this non additivity rather than being a weakness is a desirable property of the concept, at least for a large set of problems. The seemingly innocuous additivity axiom implicitly imposes too much structure on what "protocols" are feasible for the players. For example, in a standard one-buyer, two-sellers example, it implicitly imposes a restriction so that the buyer cannot attempt bundling or making joint offers for two goods when dealing with the same two potential sellers of these two goods. The $R-$solution lets the primitives of the problem speak about such possibilities. Finally, we argue that the internal consistency requirements discussed in the literature involve definitions of reduced games, whose disagreement payoffs are based on inconsistent beliefs. In contrast, the $R-$solution implicitly provides a natural definition of the reduced game based on mutually consistent beliefs.

In the main part of the paper we restrict attention to games in characteristic form, in which bilateral coalitions are mutually exclusive. In Section 5 we show how the $R-$solution can also be applied to games with coalitional externalities (partition function form games) by simply reinterpreting the value of one-player coalitions. We also explore possible routes to extend the $R-$solution to three-player games with more than one compatible bilateral coalition, and as well to games with more than three players.

[5] The nucleolus is also a selection of the BS. Thus, when the core is empty, the nucleolus and the $R-$Solution coincide. However, when the core is not empty and set-valued, the two concepts differ. We elaborate on this point in Section 4.

2 The $R-$solution of a three-player game

Let $N = \{1, 2, 3\}$ be the set of players, which can be indistinctively indexed by i, j, k, and let 2^n represent the set of subsets of N, with cardinality n. An element $Z \in 2^n$ represents a coalition. A TU game in characteristic form is the pair (N, v), where $v : 2^n \to R$ indicates the worth of each coalition and satisfies $v(\varnothing) = 0$. We assume v to be *superadditive.*

Assumption 1 (superadditivity): If $Z, Z' \subset 2^n$ and $Z \cap Z' = \varnothing$, then $v(Z) + v(Z') \leq v(Z \cup Z')$.

To save some space, we use an abbreviated notation for the v function: $v_{ij} = v(\{i, j\})$, $v_i = v(\{i\})$ and $V = v(\{1, 2, 3\})$. When we write "for all i, j" or "for all i, j, k" we mean for all $i, j = 1, 2, 3, i \neq j$, and for all $i, j, k = 1, 2, 3$, $i \neq j \neq k, i \neq k$, respectively; different sub/superindices in the same expression always denote different players. Without loss of generality, we assume that $v_{12} - v_1 - v_2 \geq v_{13} - v_1 - v_3 \geq v_{23} - v_2 - v_3$. In other words, coalition $\{1, 2\}$ is the (weakly) most "efficient" among the two-player coalitions and coalition $\{2, 3\}$ the (weakly) least efficient.

2.1 Bilateral negotiations

As discussed in the Introduction the most novel building block of our theory is the determination of disagreement payoffs for negotiations in the grand coalition based on consistent beliefs. We identify these disagreement payoffs with the expected value of outcomes in the three possible bilateral negotiations, where the expectation is taken with respect to the prediction of which of these negotiations would be implemented (with what probability). In other words, we begin by formulating a solution to the three-player/three-cake problem where players hold consistent beliefs about both what cake will be shared and how it will be shared.

Thus, consider the three possible bilateral negotiations between players i and j, for all i, j. For each player i in each bilateral negotiation (i, j), we denote i's predicted payoff by u_i^{ij}. Also, p_{ij} denotes the probability that players i and j are the ones that successfully form a coalition and implement $\left(u_i^{ij}, u_j^{ij}\right)$.[6] Finally, for each player i in each bilateral negotiation (i, j), we also define i's disagreement payoff in that bilateral negotiation, which we represent by t_i^{ij}. Before formally defining our solution, we first explain the consistency requirements on this set of values that are the basis for that solution.

i) Given the bilateral disagreement payoffs, t_i^{ij}, t_j^{ij}, players i and j share v_{ij} according to the NBS, provided this surplus is positive. That is, $u_i^{ij} = t_i^{ij} + \frac{1}{2}\left(v_{ij} - t_i^{ij} - t_j^{ij}\right) = \frac{1}{2}\left(v_{ij} + t_i^{ij} - t_j^{ij}\right)$, if $v_{ij} \geq t_i^{ij} + t_j^{ij}$. However, if their disagreement payoffs sum up to an amount in excess of the worth of the coalition, $v_{ij} < t_i^{ij} + t_j^{ij}$, then we impute $u_i^{ij} = v_i$.[7] That is, in the absence of a surplus players are not willing to reach an agreement, and payoffs in that particular negotiation coincide with one-player coalition values.

ii) Disagreement payoffs in each bilateral negotiation are endogenous, computed according to the payoffs predicted in, and the probability distribution over alternative, two-party negotiations. In particular, suppose that the negotiation between i and j flounders, and players contemplate their options within the larger picture of all two-player negotiations. As players calculate what they expect to get in this scenario, t_i^{ij}, they predict: (a) with probability p_{ij} what they face is precisely this default, t_i^{ij}; (b) with probability p_{ik} coalition (i, k) will succeed, and player i's payoff will be u_i^{ij}; and (c) with probability p_{jk} it will be the coalition (j, k) that succeeds, and hence i's payoff will be v_i. Thus, $t_i^{ij} = p_{ij}t_i^{ij} + p_{ik}u_i^{ik} + p_{jk}v_i$. If $p_{ij} < 1$ we can

[6] For completeness, we should also contemplate the possibility that no coalition would form. That is, the possibility that all three bilateral negotiations fail. Doing so would introduce no change in our solution. Consequently, we prefer to simplify the discussion by imposing the constraint, instead of obtaining it as a result, that one of the two-player negotiations will succeed in case the grand coalition did not form.

[7] We will justify this imputation below, after all the ingredients of our solution concept are laid out.

rewrite this expression as:

$$t_i^{ij} = \frac{p_{ik} u_i^{ik} + p_{jk} v_i}{1 - p_{ij}}.$$

Consequently, player i's disagreement payoff in her negotiation with j is the expected payoff in alternative negotiations, where the expectation is "conditional" on her negotiation with j having come to a halt.

iii) Finally, we place some restrictions on the set of admissible probability distributions. A coalition between players (i, j) can succeed with a non-negligible probability, p_{ij}, only if both players weakly prefer such agreement over their alternative, $u_i^{ik} \geq u_i^{ij}$ and $u_j^{jk} \geq u_j^{ij}$.

It is worth emphasizing that disagreement points of a bilateral negotiation are the payoffs that players obtain in case that particular negotiation fails but the predicted payoffs materializes in all other negotiations. Hence, disagreement points only depend on variables that are exogenous to that negotiation. Moreover, if there is not enough surplus in the negotiation between players i and k ($v_{ik} < t_i^{ik} + t_k^{ik}$) then in player i's alternative negotiation, the one between i and j, $t_i^{ij} = \frac{p_{ik} u_i^{ik} + p_{jk} v_i}{1 - p_{ij}} = v_i$. In other words, if the negotiation between players i and j fail then they anticipate that the only possible agreement is between players j and k (since i and k are not willing to form a coalition), and hence all player i can get is v_i.

For some types of games, our solution predicts that a particular coalition would form with probability one, should the three-player coalition fails to form. However, since Bayes' rule does not constrain probabilities when conditioning on zero probability events, allowing for probability-one events leaves too many degrees of freedom with respect to what are consistent outcomes in the rest of events. To avoid this indeterminacy, we proceed in the standard way of first considering only probability distributions that assign to each two-player negotiation a probability of success bounded away from 1. Thus, according to points i), ii), and iii), we begin by formally defining a prediction for simultaneous bilateral negotiations in a reduced belief space.

Definition 1 *For $\epsilon > 0$, an $\epsilon-$Prediction for Simultaneous, Bilateral Negotiations for the three-player game (N, v), $\epsilon - PSBN$ for short, is a triple $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_{ij}(\epsilon)\right\}_{i,j=1,2,3}$ that satisfies:*

1)

$$u_i^{ij}(\epsilon) = \begin{cases} \frac{1}{2}\left(v_{ij} + t_i^{ij}(\epsilon) - t_j^{ij}(\epsilon)\right) & \text{if } v_{ij} \geq t_i^{ij}(\epsilon) + t_j^{ij}(\epsilon), \\ v_i & \text{otherwise;} \end{cases}$$

2) $t_i^{ij}(\epsilon) = p_{ij}(\epsilon)\, t_i^{ij}(\epsilon) + p_{ik}(\epsilon)\, u_i^{ik}(\epsilon) + p_{jk}(\epsilon)\, v_i$, *for all i, j, k;*

3) $p_{12}(\epsilon) + p_{13}(\epsilon) + p_{23}(\epsilon) = 1$; $p_{ij}(\epsilon) \leq 1 - \epsilon$ for all i, j; and for all i, j, k, $p_{ij}(\epsilon) > \epsilon$ only if $u_i^{ij}(\epsilon) \geq u_i^{ik}(\epsilon)$ and $u_j^{ij}(\epsilon) \geq u_j^{jk}(\epsilon)$.

Our prediction for simultaneous, bilateral negotiations is the limiting value of predictions, as the upper bound on p_{ij} tends to 1.

Definition 2 *A Prediction for Simultaneous, Bilateral Negotiations for the three-player game (N, v), $PSBN$ for short, is a triple $\left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}_{i,j=1,2,3}$ such that there exists a sequence of $\epsilon-PSBN$ that converges to $\left\{u_i^{ij}, t_i^{ij}, p_{ij}\right\}_{i,j=1,2,3}$ as ϵ goes to 0.*

We simplify the presentation of our results here by normalizing $v_i = 0$ for all i, but close this section by explaining how all results and computations can be straightforwardly extended to the general case. Meanwhile, payoffs and disagreement points should be interpreted as net of the worth of one-player coalitions. Next, we state the main result of this paper.

Proposition 1 *For ϵ small an $\epsilon-PSBN$ exists for the game (N, v). Moreover, any selection of $\epsilon - PSBN$ converges to the same values as ϵ goes to 0. That is, a $PSBN$ exists and is unique, and given by:*

(Region 1) if $v_{12} \geq v_{13} + v_{23}$ and $v_{13} \leq \frac{1}{2}v_{12}$, then $p_{12} = 1$, and $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$;

(Region 2) if $v_{12} \geq v_{13} + v_{23}$ *and* $v_{13} \geq \frac{1}{2}v_{12}$*, then* $u_1^{12} = u_1^{13} = v_{13}$, $u_2^{12} = v_{12} - v_{13}$,$u_3^{13} = 0$, $p_{23} = 0$ *and if* $v_{13} < v_{12}$ *then* $p_{12} = 1$*; and*

(Region 3) if $v_{12} \leq v_{13} + v_{23}$ *then* $u_i^{ij} = u_i^{ik} \equiv u_i = \frac{v_{ij}+v_{ik}-v_{jk}}{2}$*, for all* i, j, k*, and* $p_{ij} = \frac{u_i u_j}{u_1 u_2 + u_1 u_3 + u_2 u_3}$.

Proof. See Appendix. ■

The $PSBN$ is intuitive. In Regions 1 and 2, player 3 is relatively weak. Her contribution to the coalitions with players 1 and 2 is relatively small with respect to what players 1 and 2 can otherwise obtain together. In these regions we predict that players 1 and 2 "trade" with probability one (except in the limit case of $v_{12} = v_{13}$ where these two trades are equivalent) and player 3 obtains zero. The way players 1 and 2 split the surplus depends on whether player 1 (given our notation) has a sufficiently important alternative. If player 1's alternative is relatively small (Region 1) then the payoffs of players 1 and 2 coincide with the NBS of their bilateral negotiation in isolation. Any threat by either player of turning to player 3 bears no credibility, since neither can obtain anything better outside their current negotiation. However, if player 1's potential alternative is sufficiently high, then she obtains the full value of this alternative in their negotiation with player 2. The reason is the following. In her negotiations with 2, player 1 cannot obtain anything lower than her alternative since otherwise she would have incentives to switch to player 3. She cannot obtain anything higher than her alternative either, since if that were the case the threat of turning to player 3 would not be credible.

In particular, our solution concept conforms to the "outside option principle" (see Shaked and Sutton 1984, and Binmore, Shaked, and Sutton, 1989): the payoffs of players 1 and 2 coincide with the NBS of their bilateral negotiation in isolation, unless one has an outside option that is binding. In a $PSBN$ this outside option (for a bilateral negotiation) is "endogenously" determined (by all simultaneous, bilateral negotiations).[8]

In Region 3 player 3 is relatively strong (and hence the three players are relatively similar). In this case, the $PSBN$ predicts that any of the three bilateral negotiations may succeed with positive probability. If a particular trade was predicted with probability one, then the player left out would be able to find a mutually beneficial agreement with any of the two players. This would render the prediction inconsistent. Thus, in Region 3 the only consistent prediction involves a positive probability of success for all three bilateral negotiations, which in turn requires that all players are indifferent about with whom to strike a deal. Thus, in this region efficiency fails due to the relative bargaining strength of the weakest player.

As mentioned above, the $PSBN$ is the basis for the main contribution of this paper, i.e., the construction of consistent disagreement payoffs for the negotiations inside the grand coalition. However, it is important to bear in mind that the $PSBN$ is also of independent interest. Indeed, we view the $PSBN$ as the right prediction for the three-player/three-cake game, where one bilateral trade is the only feasible outcome, and side payments between the trading partners and the non trading player are not feasible. An important economic example that closely matches this description is merger analysis. In Section 5 we discuss an example taken from Horn and Persson (2001). In our view in these cases the $PSBN$ is the right solution concept.

2.2 The trilateral negotiation

Once we have a prediction for the outcome of bilateral negotiations, we also have all we need to determine the disagreement payoffs in the three-player negotiation. This last step before defining a solution concept for the game (N, v), the $R-$solution, is straightforward: the disagreement payoff of

[8] This is another difference between the $R-$solution and the concept(s) defined in Bennett (1997). That is, by introducing coherent conjectures with respect to the probabilities of success, we endogenously obtain the "outside option principle" as a natural outcome of bargaining à la Nash.

player i in the negotiations inside the grand coalition, T_i, will simply be her expected payoff according to the $PSBN$. A formal definition is required:

Definition 3 *A $R-$solution for the three-player game in characteristic form (N,v) is a triple (U_1,U_2,U_3) that satisfies: a) $U_i=\frac{1}{3}(V+2T_i-T_j-T_k)$ for all i,j,k, b) $T_i=p_{ij}u_i^{ij}+p_{ik}u_i^{ik}+p_{jk}v_i$, where $\left\{u_i^{ij},t_i^{ij},p_{ij}\right\}_{i,j=1,2,3}$ is the $PSBN$ for the game (N,v).*

Characterizing this solution, in particular its existence and uniqueness, requires characterizing $\left\{u_i^{ij},t_i^{ij},p_{ij}\right\}$. This was done in Proposition 1. Therefore T_i, $i=1,2,3$ exists and is unique. The proof of the following theorem is straightforward.

Theorem 1 *The $R-$solution exists and is unique.*

Computing the $R-$solution is in fact extremely easy. What we offer below can be considered a user's manual. In Regions 1 and 2 defined in Proposition 1, the $PSBN$ predicts that players 1 and 2 trade with probability 1. Thus, the expected payoffs for players 1 and 2 add up to v_{12}, and player 3's payoff is zero in both regions. These payoffs are the disagreement payoffs in the three-player negotiation. Then in the $R-$solution each player obtains her disagreement payoff, plus one third of the excess of the worth of the grand coalition over the efficient bilateral coalition, $V-v_{12}$.

In Region 3 all two-player coalitions form with positive probability, in the event the grand coalition fails to reach an agreement. In this region, the $PSBN$ satisfies some interesting properties. First, as already mentioned, the payoff for a player in each of her two-player coalitions is the same: $u_i^{ij}=u_i^{ik}=u_i$.[9] Also, for each game there exists a number Ψ such that

$$p_{ij}u_k=\Psi \text{ for all } i,j,k. \tag{1}$$

[9]This number coincides with what Davis and Maschler (1965) called i's quota. That is, i's payoff in the *kernel for a coalition structure* $\{(i,j),k\}$.

It is important to note that p_{ij} is the probability that player k does not get u_k. Therefore, condition (1) indicates that the "loss" experienced by player k with respect to the benchmark where she is able to secure u_k with probability one, is identical for all $k = 1, 2, 3$. This property drastically simplifies the computation of final payoffs. More specifically, player i's expected payoff in the $PSBN$ is:

$$T_i = (p_{ij} + p_{ik}) u_i = u_i - \Psi,$$

where we have used (1) to obtain the last equality. As a result player i's payoff in the $R-$solution is:

$$U_i = \frac{1}{3}(V + 2T_i - T_j - T_k) = \frac{1}{3}(V + 2u_i - u_j - u_k).$$

This makes it possible to compute the $R-$solution without computing the probabilities p_{ij}. Using Proposition 1 we obtain the final expression:

$$U_i = \frac{1}{3}(V + v_{ij} + v_{ik} - 2v_{jk}).$$

When $v_i \neq 0$, all these computations involve merely substituting $v_{ij} - v_i - v_j$ for v_{ij}, for all i, j, and $V - v_1 - v_2 - v_3$ for V, and also adding v_i to all values u, t, U and T. Table 1 contains the expression for the $R-$solution in this general case.

Table 1: The $R-$solution

	Region 1	Region 2	Region 3
U_1	$\frac{2V+v_{12}+3v_1-3v_2-2v_3}{6}$	$v_{13} - \frac{4v_3}{3} + \frac{V-v_{12}}{3}$	$\frac{V+v_{12}+v_{13}-2v_{23}}{3}$
U_2	$\frac{2V+v_{12}-3v_1+3v_2-2v_3}{6}$	$\frac{2(v_{12}-v_{13})}{3} + \frac{2}{3}v_3 + \frac{V-v_{13}}{3}$	$\frac{V+v_{12}+v_{23}-2v_{13}}{3}$
U_3	$v_3 + \frac{V-v_{12}-v_3}{3}$	$\frac{2}{3}v_3 + \frac{V-v_{12}}{3}$	$\frac{V+v_{13}+v_{23}-2v_{12}}{3}$

Consider the case $V = v_{12} + v_3$, where the best that the grand coalition can do is to implement the "efficient" bilateral trade. It turns out that, in Regions 1 and 2, the grand coalition is redundant: $U_i = T_i$. (In particular, $U_3 = v_3$.) However, in Region 3 the grand coalition can fix the bargaining failure implicitly identified by the $PSBN$. It is noteworthy that in this region the non-participating player 3 is able to appropriate a positive surplus ($U_3 > v_3$), precisely because players 1 and 2 are willing to "bribe" player 3 in exchange for not interfering with the implementation of the efficient bilateral trade.[10]

3 A non-cooperative implementation

This section presents and solves a non-cooperative bargaining protocol that asymptotically implements the $R-$solution defined in the previous section. The goal is not only to show that the solution is the outcome of *some* game, but also to illustrate the implications of our consistency requirement in terms of bargaining protocol flexibility.

Consider the following infinite horizon, discrete-time, $t = 1, 2, 3, ...,$ game $\Gamma(\delta, \alpha)$ played by the three players. At the beginning of each period, the game may be in either a three-player negotiations phase or a bilateral-negotiations phase. In period 1, the game begins in the three-player negotiation phase. If in period t the game is in a three-player negotiation phase and does not finish in time t, then in period $t + 1$ the game moves to a bilateral-negotiations phase with some (perhaps minuscule) probability α. Otherwise, it continues in the three-player negotiation phase. If the game enters a bilateral-negotiations phase at time $t + 1$, then it will stay in a bilateral-negotiations phase for all $t' > t + 1$ until it ends. Thus, we may interpret α as the exogenous probability that three-player negotiations break down. If they do, then there will never be trilateral negotiations again, although players can keep attempting bilateral agreements.

The stage game in the three-player negotiation phase begins with a move

[10] In games where $V = v_{12} + v_3$ the Shapley value still grants player 3 a positive payoff, even though not directly participating in the formation of the efficient coalition. A standard justification is that player 3's positive payoff is precisely the bribe for not interfering with the efficient outcome. We partly use the same intuition, but only apply it in the case player 3 has the actual power of preventing the formation of the efficient coalition (Region 3).

by Nature, who chooses with equal probability one of the players. The chosen player then makes a proposal $s \in \mathbf{R}^3$ to divide V. After the proposal is (publicly) made, the other two players accept or reject in turn, say by the order of the integers. In case of acceptance by both, the game ends and the payoffs are s_i according to the proposal. In the event either or both players reject the proposal, the game moves to the next period.

If the game is in the bilateral-negotiations phase, so that it will be in that phase forever, the stage game is as follows:

(1) Nature selects one of the three players with equal probability. Let that player be A.

(2) Player A invites one of the other two players to become her negotiation partner. Let us call her player B.

(3) Player B accepts or rejects. If she accepts then players (A, B) move to step (4). If player B rejects then players (B, C) move to step (4).

(4) Let (E, F) be the two players that move to this step from step (3). Nature selects one of them with equal probability. Let the chosen one be F.

(5) Player F makes a proposal to player E to share v_{EF}. Let that proposal be θ_F^E, understood as the share of v_{EF} that F keeps.

(6) Player E accepts or rejects F's proposal. If E accepts then she gets $v_{EF} - \theta_F^E$, while player F gets her demand θ_F^E and the game ends. The third player obtains nothing. If E rejects the offer then the game moves to the next period.

Thus, in the stage game of the bilateral negotiations phase players first engage in a process that selects the negotiating partners (steps (1)-(4)), and next the pair of selected players conduct the actual negotiation (steps (5)-(6).)

Players discount the future at the common discount factor, $\delta \in (0, 1)$. We look for subgame perfect equilibria in stationary strategies for this game.

First, however, let us look closely at the characteristics of this protocol.

Assume $\alpha = 0$. The three-player negotiation phase is the (projection of the) simplest, standard protocol that implements the multilateral bargaining solution for pure bargaining problems with exogenous disagreement payoffs, as $\delta \longrightarrow 1$. What we add to this protocol is a (perhaps very small) probability that the three-player negotiations break down, in which case players remain free to try to attempt bilateral agreements. Once the pair of negotiating players have been selected, they can attempt to find a mutually profitable agreement again following the simplest protocol that, in isolation, would implement the bilateral Nash bargaining solution.

Only two features of this protocol require some comments. The first is relatively straightforward: we introduce an exogenous probability that three-party negotiations break down. In bargaining protocols, the discount factor typically has two equivalent, alternative interpretations. One is impatience in the face of delay, which is what we put to work here. The other is (one minus the) probability that negotiations break down. Here we could also use this second interpretation. In fact, if we substitute some exogenous disagreement payoffs for the bilateral-negotiations phase when three-party negotiations break, we (could set $\delta = 1$, for instance and) obtain the multilateral Nash bargaining solution for those exogenous disagreement payoffs when α approaches 0. Actually, we show this property and use it as an intermediate step.

The second feature of this protocol that deserves some discussion is the way that parties to a bilateral negotiation are chosen -in particular steps (2) and (3) of the bilateral-negotiations phase. These steps are the most distinctive features of our protocol: their effect is to allow the three players full flexibility as to which bilateral negotiation is attempted in each period of time. In our approach there is no implicit imposition of any lower or upper bound on the probability of any agreement or any set of agreements. Nature's movement does not allow any advantage. In particular, suppose that along an equilibrium path players i and j prefer meeting with each other rather than meeting with player k, although they still prefer an agreement with k rather than waiting, and taking a chance in the next period. Then, even in the most favorable realization (Nature chooses her in step (1)), there is nothing that player k can do to prevent players i and j from meeting in the current period. This would not be the case if we used more standard protocols, where either Nature or an exogenous rule selects who is a party to the bilateral negotiation in a given period. For instance, in Fridolfsson and Stennek (2005)[11] the player chosen by Nature in step (1) makes an offer to the player she selects in step (2), thus jumping directly to step (5). That scenario implicitly imposes arbitrary restrictions on the set of feasible payoffs. For instance, in any equilibrium with no delay (where an agreement is reached in the first period with probability one) each player, even the weakest (player 3 in our case), has a probability of at least one third of being part of the successful coalition. Thus, due to the protocol the expected payoff of the weakest player is artificially enhanced. This kind of lower bound in the probability of being part of a successful coalition is absent in our protocol, wherein players' choices determine the outcome of the negotiations with no exogenous constraint other than the primitives of the problem.

We now turn to exploring the equilibrium of this game. Note that all subgames following Nature's choice to enter into the bilateral-negotiations phase will be the dynamic game with stage game described by steps (1) to (6) above. Denote that game by $\Phi(\delta)$. If $\Phi(\delta)$ had unique (subgame perfect, stationary) equilibrium payoffs, $\mathbf{x} \in \mathbf{R}^3$, then all (subgame perfect, stationary) equilibria of the whole game $\Gamma(\delta, \alpha)$ would coincide with the equilibria of the game $\widehat{\Gamma}(\delta, \alpha; \mathbf{x})$, obtained by substituting a terminal node

[11] Also, see Binmore (1985) and Binmore, Shaked, and Sutton (1989) for surveys of other protocols for the three-player/three-cake problem. Ray (2007) offers, among other intersting things, a comprehensive discussion of general features of protocols for coalition formation.

with payoffs $\mathbf{x}$ for each node where Nature chooses to switch to a bilateral-negotiations phase. As indicated before, showing that $\widehat{\Gamma}(\delta, \alpha; \mathbf{x})$ implements the multilateral Nash bargaining solution with disagreement payoffs $\mathbf{x}$ as $\delta \to 1$ (plus the necessary continuity of the equilibrium correspondences) is rather straightforward. We do this in Proposition 3 below, but the main result here is that equilibrium payoffs of $\Phi(\delta)$ approach $\mathbf{T} \in \mathbf{R}^3$ as $\delta \to 1$, where $\mathbf{T}$ is the vector of expected payoffs of the $PSBN$. That is, the main result is that $\Phi(\delta)$ implements the $PSBN$ as δ goes to 1. This result is interesting in itself. Indeed, it offers a non cooperative implementation of our solution to the three-player/three-cake problem.

We need to introduce some additional notation to analyze the game $\Phi(\delta)$. A stationary strategy for player i may be described by $\left(\mu_i^j, \lambda_i^j, \lambda_i^k\right)$ for steps (2) and (3), and $\left(\theta_i^j, \rho_i^j, \theta_i^k, \rho_i^k\right)$ for the remaining steps. $\mu_i^j \in [0,1]$ is the probability that player i proposes player j to be her negotiation partner in step (2), if i is chosen by Nature in step (1). Given the definition of the game, the probability that i proposes k is $\mu_i^k = 1- \mu_i^j$. $\lambda_i^j \in [0,1]$ is the probability that player i accepts player j's invitation to become a negotiation partner in step (3), if so proposed, and λ_i^k is the probability i accepts player k's invitation. In line with the restriction to stationary strategies, we impose the constraint that $\lambda_i^j = 1 - \lambda_i^k$. That is, player i accepts an invitation to meet player j with the same probability that she rejects player k's invitation (and instead meet player j). Thus, in case Nature chooses player i, the probability that players (i,j) negotiate in steps (5) and (6) is $\mu_i^j \lambda_j^i$; the probability that (i,k) negotiate is $\mu_i^k \lambda_k^i = \left(1 - \mu_i^j\right) \lambda_k^i$; and the probability that (j,k) negotiate is $\mu_i^j \lambda_j^k + \mu_i^k \lambda_k^j = \mu_i^j \left(1 - \lambda_j^i\right) + \left(1 - \mu_i^j\right)\left(1 - \lambda_k^i\right)$.

We implicitly rule out the possibility that player i makes no invitation to meet when Nature chooses her to be the first proposer. This again is in line with the spirit that no player can veto other players' meeting: by assuming that player i has to make an invitation, and so has to give either player j or player k a chance to decide on the pair that meets, we prevent her from vetoing a meeting between players j and k. Also, we assume that if a player is offered a first invitation to meet then she either accepts the invitation or meets the third player. Similarly, we assume in the latter case that meeting will take place, and the third player cannot refuse to meet. The reader should note this is only to simplify the moves, since meeting a partner and not making or accepting offers is always a choice and essentially amounting to refusing meeting at all.

If in node (4) player i is chosen by Nature as the proponent (and player j as the respondent), then in node (5) with probability $\rho_i^j \in [0, 1]$ she offers player j a payoff equal to $v_{ij} - \theta_i^j$. θ_i^k and ρ_i^k are the corresponding values in a negotiation with k. In order to avoid open-set technical problems, and also save in notation, we assume that in node (6) the respondent accepts with probability one any offer above or equal to her continuation value. That is why we have chosen to not include these decisions in the description of a strategy. As we will show below, this is innocuous and in particular does not rule out the possibility of delay in case of indifference.[12] Again, note that in line with the restriction to stationary strategies, we implicitly assume the answer to invitations to negotiate in node (3) and the offer in node (5) do not depend on who made the invitation to meet or who answered to that invitation, but rather only on the identity of the partners in the meeting.

The next result concerns the non-cooperative implementation of the $PSBN$.

Proposition 2 *For δ sufficiently close to 1, a stationary, subgame perfect equilibrium of $\Phi(\delta)$ exists, in which an agreement is reached in the first period. Moreover, in any selection of stationary, subgame perfect equilibria,*

[12]Indeed, apart from open-set issues, in a SPE there could be indifference between accepting and rejecting a partner's offer only if the sum of the continuation values for both partners is *equal* to what they have to share. In this case, the fact that the proponent can choose any value ρ in $[0, 1]$ already allows for any probability of delay.

the probability that the game ends in agreement between players i and j converges to p_{ij} of the $PSBN$ for any i, j, and expected payoffs also converge to the payoffs, T, of the $PSBN$ as δ goes to 1.

The proof can be found in the Appendix.

As we advanced, this proposition is the main result of this section. In particular, it determines that in any stationary SPE, the payoffs in the bilateral-negotiation phase of $\Gamma(\delta, \alpha)$, as $\delta \to 1$, are well defined. Moreover, it shows that $\Phi(\delta)$ implements the $PSBN$ as a solution to the three-player/three-cake problem.

Remark 1 *As $\delta \longrightarrow 1$, $\Phi(\delta)$ implements the $PSBN$.*

Once we are provided with this result, it is straightforward to show that $\Gamma(\delta, \alpha)$ implements the $R-$solution as $\delta \to 1$. We do so by analyzing the game $\widehat{\Gamma}(\delta, \alpha; \mathbf{x})$ for arbitrary $\mathbf{x} \in \mathbf{R}^3$. Naturally, we restrict attention to $\mathbf{x}$ so that $x_1 + x_2 + x_3 \leq V$.

Proposition 3 *For any α, there is a unique stationary, subgame perfect equilibrium of $\widehat{\Gamma}(\delta, \alpha; \mathbf{x})$ where players reach an agreement in the first period and equilibrium payoffs are continuous for $\delta, \alpha > 0$. Moreover, as δ goes to 1, equilibrium payoffs converge to the NBS of the tree-player game with disagreement payoffs $\mathbf{x}$.*

The proof for Proposition 3 can be found in the Appendix. Thus, as $\delta \longrightarrow 1$ equilbrium payoffs of $\Phi(\delta)$ are arbitrarily close to the expected payoffs of the $PSBN$, $\mathbf{T}$, and the unique equilibrium of $\widehat{\Gamma}(\delta, \alpha; \mathbf{x} = \mathbf{T})$ is arbitrarily close to the $R-$solution. Since this latter equilibrium is continuous in $\mathbf{x}$, we conclude that as $\delta \longrightarrow 1$ any subgame perfect, stationary equilibrium of $\Gamma(\delta, \alpha)$ is arbitrarily close to the $R-$solution.

Corollary 1 *As $\delta \longrightarrow 1$, $\Gamma(\delta, \alpha)$ implements the $R-$solution for any α.*

Note that α can be as small as desired. It may even vanish in the limit, as long as it does so at a speed less than the speed at which $\delta \longrightarrow 1$.

Thus, Propositions 2 and 3 show that a simple, yet flexible bargaining protocol exists whose asymptotitc outcome is as predicted by the $R-$solution.

Although we take a different approach and focus on another set of questions, some of our results are roughly consistent with those of Compte and Jehiel (2010). In particular, if we restrict ourselves to the class of games with $V = v_{12}$, and players start the game in the bilateral negotiations phase, then our protocol implements and efficient outcome only if the core is not empty (the game lies in Regions 1 or 2). This is in line with their main result. On top of that we identify which sort of inefficiency we should expect otherwise. When we consider the entire game, the feature that generates an efficient outcome, even for games with an empty core, is the possibility that trilateral negotiations may irreversibly come to and end.[13]

4 Properties of the $R-$solution

In this section we study the properties of the $R-$solution by discussing its relation with the two main single-valued solution concepts in cooperative game theory, its inclusion relation to set value concepts and geometrical properties, and then the internal consistency rationale behind its interpretation as a generalization of the NBS.

[13] This feature does not "impose" that all coalitions are relevant in the game, as Compte and Jehiel (2010) argue that other protocols in the literature do. In particular, coalition (2,3) is irrelevant in Regions 1 and 2, and coalition (1,3) is irrelevant in Region 1. Thus, the $PSBN$ and the protocol for bilateral negotiations do satisfy the outside option principle. However, unlike Compte and Jehiel (2010), for games in which $V > \frac{2}{3}v_{12}$ the $R-$solution and the entire bargaining protocol does not satisfy the outside option principle, since bilateral coalitions do matter. The reason is that, in the spirit of Harsanyi (1959), our concept and bargaining protocol treat the grand coalition differently from the rest.

4.1 The $R-$solution, the core, and the bargaining set

We are mainly interested in the relationship between the core and the bargaining set (of the grand coalition), BS. The following simple lemma simplifies the ensuing discussion.

Lemma 1 *For three-player, superadditive TU-games, the bargaining set of the grand coalition coincides with the core if the latter is not empty. If the core is empty, then the bargaining set of the grand coalition is a singleton.*

The proof of this *folk* lemma is given in the Appendix. This lemma allows us to consider only the relationship between the $R-$solution and the BS.

Proposition 4 *The $R-$solution belongs to the bargaining set (for the grand coalition) and so to the core if the latter is not empty.*

Proof. First, we study the core. An element of the core is a positive vector (x_1, x_2, x_3) such that: (i) $x_1 + x_2 + x_3 = V$, (ii) $x_i + x_j \geq v_{ij}$ for all i, j, and (iii) $x_i \geq 0$, for all i. Adding the three conditions (ii), we obtain $x_1 + x_2 + x_3 \geq \frac{v_{12}+v_{13}+v_{23}}{2}$, which combined with condition (i) gives the condition for the core not to be empty:

$$V \geq \frac{v_{12} + v_{13} + v_{23}}{2}. \tag{2}$$

When $v_{12} \geq v_{13} + v_{23}$, i.e., in Regions 1 and 2, this is satisfied. It is then immediate to check that the $R-$solution satisfies (i), (ii), and (iii) in Regions 1 and 2, so that in these regions the $R-$solution belongs to the core and then to the BS. In Region 3 the core may be empty; i.e., (2) may not hold. Thus, we will show that the $R-$solution belongs to the BS. We already know that in Region 3 $U_i = \frac{V+v_{ij}+v_{ik}-2v_{jk}}{3}$. Since $U_i \geq 0$ for all i, and since the grand coalition cannot be part of an objection, we need only consider objections that use two-player coalitions. Thus, consider an objection of i against j.

That is, consider a division of v_{ik}, $x = (x_i, x_k)$ where $x_i + x_k = v_{ik}$, such that $x_i > U_i$, and $x_k > U_k$. We show that there is a counter-objection of j, that is, a division $y = (y_j, y_k)$ of v_{jk} where $y_j + y_k = v_{jk}$, such that $y_j \geq U_j$ and $y_k \geq x_k$. Consider in particular $y_j = U_j$, so that $y_k = v_{jk} - U_j$. If $x_i > U_i$, then $x_k = v_{ik} - x_i < v_{ik} - U_i$. But then

$$y_k - x_k > v_{jk} - U_j - (v_{ik} - U_i) = 0.$$

Thus, if x is an objection then y is a counter-objection. QED. ■

The core is always non empty in Regions 1 and 2. In Region 3, when $V < \frac{v_{12}+v_{13}+v_{23}}{2}$, the core is empty and the BS is a singleton. Since the $R-$solution belongs to the BS, we conclude that the $R-$solution *coincides* with the BS, and as well with any selection or subset of the BS.

The $R-$solution satisfies some interesting geometric properties, in particular when $V = v_{12}$. That is, when the three-person game simply consists of three alternative bilateral trades.

Proposition 5 *If $V = v_{12}$, the $R-$solution coincides with the selection of the most egalitarian allocation in the bargaining set. Thus, it also coincides with the selection of the most egalitarian allocation in the core, when the core is not empty.*

Proof. Note that $U_1 \geq U_2 \geq U_3$. Thus, a more egalitarian allocation would require increasing the payoff of player 3 or, at least, increasing the payoff of player 2 by reducing the payoff of player 1. We show first that in Region 1 and Region 2 any allocation x in the BS or, equivalently in these regions, in the core assigns a payoff $x_3 = 0$. Assume otherwise $x_3 > 0$. Then $x_1+x_2 = v_{12}-x_3 < v_{12}$, so that the allocation would not be in the core. This immediately proves that the $R-$solution is the most egalitarian allocation in the BS for Region 1. Now suppose we are in Region 2 and that there is an allocation x which is more egalitarian than the $R-$solution. Since $x_3 = 0$, this implies that $x_2 > v_{12} - v_{13}$, so that $x_1 + x_3 = v_{12} - x_2 < v_{13}$, thus violating the conditions for x to be in the core. The $R-$solution is clearly the most egalitarian allocation in the BS in Region 2. Finally, in Region 3 the core is empty, so that the BS is a singleton. Thus, the $R-$solution is the *only* allocation in the BS. QED ■

Therefore, the $R-$solution is the most egalitarian among the *stable* (in the sense of Aumman-Maschler) allocations. That is, the most egalitarian among the allocations that cannot be blocked in the sense of the (grand coalition) BS. When $V > v_{12}$, the $R-$solution "distributes" the excess $V - v_{12}$ equally among all players. In that sense, the $R-$solution treats all players as equally as is compatible with their relative strength in bilateral negotiations, but does not compensate weaker players when the grand coalition adds surplus.[14]

4.2 The $R-$solution, the Shapley value, and the nucleolus

It is well known that the Shapley value or any other probabilistic value is not necessarily in the core or the BS.[15] This is the first difference between the $R-$solution and the Shapley value. In fact, they coincide only at two points of the parameter space: $v_{13} = v_{23} = 0$ and $v_{13} = v_{23} = v_{12}$.[16] For the rest of the parameter space, the comparison is straightforward and some regularities can be noticed. With respect to the Shapley value, according to the $R-$solution: (i) Player 3's payoff is always lower, (ii) Player 2's payoff is lower if and only if v_{13} is sufficiently high, (iii) Player 1's payoff is lower

[14] Note that, in regions 1 and 2 and for $V = v_{12}$, the disagreement point T is on the efficient frontier, and so coincides with the $R-$solution. In Region 3 this is not the case, but the $R-$solution for the general case $V > v_{12}$ lies on the segment connecting T with the $R-$solution for $V = v_{12}$.

[15] As shown by Weber (1988), a probabilistic value is efficient only if it is a *random-order value*, and in our superadditive setting efficiency is a condition for an allocation to be in the core. The set of all random-order values contains the core, but no single one is "always" contained in the core, even if we restrict attention to three player games.

[16] In the first point ($v_{13} = v_{23} = 0$) the $R-$solution coincides with the NBS of the game for players 1 and 2. In this sense, both the Shapley value and the $R-$solution are generalizations of the NBS for three-player pure bargaining games.

if and only if both v_{13} and v_{23} are sufficiently small.[17]

A further difference between the two concepts is well illustrated by reference to the standard axioms in cooperative game theory. Indeed, the Shapley value can be characterized as the only value that satisfies the axioms of efficiency, symmetry, dummy player, and additivity (see for instance Winter, 2002). It is straightforward to check that the $R-$solution also satisfies efficiency, symmetry, and the dummy axiom. Therefore, the $R-$solution must violate the additivity axiom. Formally, if (N, v) and (N, v') are two games with solutions U and U' respectively, and we consider the game (N, v'') where $v''(Z) = v(Z) + v'(Z)$ for all $Z \subset N$, it may be that its $R-$solution U'' does not satisfy $U_i'' = U_i + U_i'$.

We will argue that when interested in simultaneous, bilateral negotiations, such violation is a strength of the concept rather that a weakness.

Example 1 *One buyer, B, can trade with two potential sellers, S and E. There are two goods and the buyer demands one unit of each. In the production of the first, S has a cost advantage, so that $v(B, S) = 1$ and $v(B, E) = \alpha \in \left(\frac{1}{2}, 1\right)$, whereas in the production of the second it is E who has the cost advantage, so that $v'(B, E) = 1$ and $v'(B, S) = \alpha$. According to the $R-$solution, E obtains 0 in the first game and $1 - \alpha$, in the second. The game $v'' = v + v'$ satisfies $v''(B, E) = v''(B, S) = 1 + \alpha$, and $v(B, S, E) = 2$.*

Additivity implies that player E, for instance, should still fetch $1 - \alpha$ in game v''. On the contrary, in v'' the $R-$solution predicts that her payoff is one third of that amount. Note that additivity amounts to assuming that the negotiations over the two goods are conducted independently. Thus, by imposing additivity, as the Shapley value does, we would be implicitly allowing sellers to commit to negotiate over each of the two goods only

[17] This discussion implies that the outcome predicted by the Shapley value does not always Lorentz dominate the outcome predicted by the $R-$solution. However, if we compute, say the variance of the outcomes, the Shapley value is always less disperse than the $R-$solution. In this sense, the Shapley value is more egalitarian.

through independent agents. These agents would not listen to anything related to the other good. The $R-$solution does not presume any ability of any party to preclude the two negotiations to interact, and so does not assume such commitment power for any player.[18],[19]

The other main single-value solution concept is the nucleolus. Like the $R-$solution, the nucleolus is a selection of the BS, and as such also belongs to the core, when the latter is not empty. Moreover, the nucleolus is also a selection of the kernel, itself a subset of the BS. Thus, by Lemma 1, when the core is empty the four concepts, BS, nucleolus, kernel, and $R-$solution, coincide. When the core is not empty, however, the two concepts are in general different. Assume $v_{12} = V$, and assume that we are in Regions 1 or 2, where the core is not empty. The nucleolus is $(x_1, x_2, x_3) = \left(\frac{1}{2}(v_{12} + v_{13} - v_{23}), \frac{1}{2}(v_{12} + v_{23} - v_{13}), 0\right)$.[20] This differs from the $R-$solution, and in particular, as Proposition 5 states, it is less egalitarian than the $R-$solution. That is, in these cases the nucleolus treats player 3 as the $R-$solution, but it treats the stronger player 1, more favorably.

[18] In the previous example, the reader may conclude that the fact that S may also supply the good for which it has a competitive disadvantage is a handicap. This is not the case. Consider the game $\widetilde{v} = v''$ except that $\widetilde{v}(B, S) = 1$. In this case, S's payoff is still a third of $1 - \alpha$ in the $R-$Solution. Thus, the $R-$solution, contrary to the Shapley value, implicitly postulates that S has no commitment device stronger than simply this sort of "burning the ships": destroying one's ability to deliver what is not going to be delivered. As this example shows, this is in particular not stronger but weaker than schemes like delegation to independent agents.

[19] Young (1985) offers another well known axiomatization of the Shapley value. Young proves that the Shapley value is the only efficient and symmetric solution that satisfies monotonicity. Monotonicity requires that, if i's marginal contribution to every coalition in a game (N, v) is higher than in another game (N, v') then i's payoff is higher in the former. In the one buyer, two seller example, monotonicity means that if, say, a technical innovation reduces one of the seller's cost, this seller's payoff should increase no matter how the innovation affects the cost of the rival seller. That is, even if the cost of the rival is reduced much more. Evidently, this is something we would not like to impose on our prediction for the example.

[20] See Leng and Parlar, 2010.

4.3 Consistency and the NBS

The relationship between the Shapley value, the nucleolus (or kernel, in the games we are considering) and the $R-$solution, on the one hand, and the NBS, on the other hand, is perhaps best illustrated in the context of the internal consistency requirement first invoked by Harsanyi (1959). Limiting the discussion to three player games, internal consistency requires that, if $x = (x_1, x_2, x_3)$ is the solution of the game (N, v), then the solution to the *reduced game* for any pair of players $\{i, j\}$ is precisely (x_i, x_j). Here the reduced game for players $\{i, j\}$ should be interpreted as a two-player game $(\widehat{N}, \widehat{v}^{ij})$ where $\widehat{N} = (i, j)$, and $\widehat{v}^{ij}(\widehat{N}) = x_i + x_j$. Now, different concepts of internal consistency may be defined that differ in the way the values $\widehat{v}^{ij}(\{i\})$ and $\widehat{v}^{ij}(\{j\})$ are chosen. Harsanyi (1959) himself, and later Hart and Mas-Colell (1989) have shown that the Shapley value satisfies internal consistency if these values are chosen in a particular way; and Sobolev (1975) and Peleg (1986) have proved the same for the nucleolus and the kernel if they are chosen in a different way.

A solution concept satisfying internal consistency so that (x_i, x_j) is the NBS of $(\widehat{N}, \widehat{v}^{ij})$ may be regarded as a generalization of the NBS. Thus, accepting Harsanyi's internal consistency requirement, the discussion about the right generalization of the NBS boils down to a discussion of the appropriate values $\widehat{v}^{ij}(\{i\})$ and $\widehat{v}^{ij}(\{j\})$ as disagreement payoffs in the negotiation of i and j. We argue that both proposals mentioned could only be generated by inconsistent beliefs, just like in the solution concepts proposed by Bennett (1997). Let us focus on the simple case where $V = v_{12}$, and $v_i = 0$ for all i. Peleg (1986) proposes that $\widehat{v}^{ij}(\{i\}) = v_{ik} - x_k$, and $\widehat{v}^{ij}(\{j\}) = v_{jk} - x_k$. Thus, in such a reduced game players i and j negotiate about the division of $x_i + x_j$, using as disagreement payoffs $v_{ik} - x_k$ and $v_{jk} - x_k$. This amounts to both players i and j expecting that if their negotiation breaks down they will be able to bribe player k to "trade" with them by offering her what she would get in the grand coalition, and keep the rest of what the two-player coalition could make. These disagreement payoffs are mutually incompatible, since players i and j cannot both form a coalition with player k. In that sense, the beliefs that sustain these disagreement payoffs are inconsistent.

The same conclusion applies to the internal consistency criterion used by Hart and Mas-Colell (1989). Their alternative definition of the reduced game has $\widehat{v}^{ij}(\{i\}) = v_{ik} - \frac{v_{ik}}{2}$ and $\widehat{v}^{ij}(\{j\}) = v_{jk} - \frac{v_{jk}}{2}$. Once again, this amounts to players i and j each expecting an agreement with player k (with probability one). The only difference is that in this case, player k is not expected to obtain x_k, but the NBS of the two-player game in isolation. That is, players i and j expect to obtain $\frac{v_{ik}}{2}$ and $\frac{v_{jk}}{2}$ when they form a coalition with player k, as the alternative to their own agreement in the reduced game.

The $R-$solution is also internally consistent when defining disagreement points as $\widehat{v}^{ij}(\{i\}) = T_i$.[21] Moreover, these disagreement points are the result of consistent beliefs. Indeed, what motivates these disagreement payoffs is the belief, of all players, that the threat for each player that disagrees with the division of surplus in the grand coalition is to revert to (simultaneous) bilateral negotiations. All players share beliefs about the outcomes of such bilateral negotiations.

Are there ways to reconcile the more standard solution concepts, Shapley value and nucleolus, with internal consistency and consistent beliefs of this sort? Answering this question positively requires us to specify $\widehat{v}^{ij}(\{j\})$ as the expectation of outcomes compatible with the coalition structure of the game, taken with respect to a well specified probability distribution over such outcomes. We claim this is an exercise that may shed new light of such important solution concepts, and so is worth exploring.

[21] Although we have defined the $R-$solution only for three-player games, it is clear the analogous solution for two-player games is the NBS.

5 Concluding remarks

In the previous sections we have restricted attention to games in characteristic form, in which only one bilateral coalition can form. While some economic models fit in such a set of games perfectly,[22] others do not. At this point is vital to look ahead and discuss how to extend the $R-$solution to a broader class of games. We first discuss an example due to Horn and Persson (2001), where bilateral agreements generate externalities. We argue the $R-$solution can also be applied to this type of games (partition function form) by simply reinterpreting v_i. Moreover, in this example the grand coalition cannot form and hence in this case the natural solution concept is the $PSBN$. Next, and in the context of an example due to Hart and Moore (1990), we show how to extend the $R-$solution to cases where bilateral trades are not mutually exclusive. Finally, we briefly discuss the kinds of problems that arise in the study of games with more than three players.

5.1 Coalitional externalities

Horn and Persson (2001), HP, present a model of endogenous merger formation. Here we focus on their example (Section 2.1) that posits a market initially populated by three oligopolistic firms. Any pair can merge but full monopolization is prohibited. Although firms are symmetric in the status quo, the synergies generated by alternative mergers are asymmetric. Firms' profits in the no merger case are normalized to 0. Profits of the firm resulting from the merger between firms i and j, and the non-merged firm k, are denoted by π_{ij} and π_k, respectively. They are:

$$\pi_{12} = 70, \pi_3 = 50,$$

$$\pi_{13} = 100, \pi_2 = 0,$$

[22] One particular example is the analysis of exclusive contracts conducted in Segal and Whinston (2000). In a previous version of this paper we discussed how the application of the $R-$solution alters the conclusions of this model.

$$\pi_{23} = 90, \pi_1 = 5.$$

In previous sections we defined the $R-$solution for games in characteristic form, where the value of a coalition is independent of the agreements reached by players not included in the coalition. However, in HP the value of one-player coalitions does depend on whether or not the other two players actually form a coalition (merge). Thus, this model can be described as a game in partition function form (Lucas and Thrall, 1963). In a three-player game, we need to specify what player i can obtain if no coalition is formed, $w^i_{\{\{i\},\{j\},\{k\}\}}$, and what player i obtains if the other two players do form a coalition, $w^i_{\{\{i\},\{j,k\}\}}$. The definition of the $R-$solution already takes into account possible externalities. The value of an individual coalition plays a role only in the definition of the disagreement payoffs in bilateral negotiations, t^{ij}_i and t^{ik}_i. These values are obtained as a probability distribution over the events that can be expected as an alternative to i forming a coalition with j or k, respectively. The only such event that has i standing alone is the formation of coalition $\{j,k\}$. Thus, only $w^i_{\{\{i\},\{j,k\}\}}$ matters, and then v_i should be interpreted as this value.[23] Summarizing, the $PSBN$ and the $R-$solution defined for games in characteristic form can also be applied to games in partition function form, simply by substituting the worth of individual coalitions, v_i, with the worth of individual coalitions conditional on the other two players forming a coalition, $w^i_{\{\{i\},\{j,k\}\}}$.

As recognized by HP, in this context side payments between merging firms and non-merging firms are unlikely to be feasible. Thus, the appropriate solution concept for this game is not the $R-$solution but the $PSBN$. In HP's example, the net surplus created by each merger is:

$$\pi_{12} - \pi_1 - \pi_2 = 65,$$

$$\pi_{13} - \pi_1 - \pi_3 = 45,$$

[23] Myerson (1977) extended the Shapley value for partition function form games. In his extension, player i's payoff depends on both $w^i_{\{\{i\},\{j\},\{k\}\}}$ and $w^i_{\{\{i\},\{j,k\}\}}$.

$$\pi_{23} - \pi_2 - \pi_3 = 35.$$

Thus, the most efficient merger (from the point of view of firms' profits) is the one between firms 1 and 2, and the game lying in Region 3. As a result, and in contrast to the conclusion reached by HP, there is a probability that an inefficient merger takes place. In fact, conditional on being part of the successful coalition, and according to the $PSBN$, players obtain: $u_1 = 30$, $u_2 = 40$, $u_3 = 60$, and the probability that the merger between 1 and 2 is successful is given by:

$$p_{12} = \frac{(u_1 - \pi_1)(u_2 - \pi_2)}{\sum_{i<j,i,j=1}^{3} (u_i - \pi_i)(u_j - \pi_j)} = \frac{21}{34}.$$

5.2 Non-exclusive bilateral trades

Hart and Moore (1990), HM, study how the allocation of property rights over assets affects the ex-post relative bargaining position of different players, which in turn determines ex-ante incentives to undertake asset-specific investments. In their famous introductory example, there are three players, the tycoon, the skipper and the chef, and one asset, the yacht. The chef is able to offer a service in the yacht worth x to the tycoon. Similarly, the skipper is able to offer a service in the yacht worth y to the tycoon. Suppose that x is the deterministic result of costly investment, but y is an exogenous parameter. Hence, there are two possible trades and they are compatible (the tycoon can enjoy a dinner in the yacht and sailing simultaneously). The game is composed of three stages. In the first, property rights over the yacht are allocated. The value of these two potential trades can only be realized if the yacht is owned by one of the players engaged in the trade. In the second stage, the chef chooses x, and in the third, given the values of (x, y), players engage in bargaining. The allocation of property rights that optimize investment incentives will depend on the predicted outcome of the bargaining game.

The $R-$solution (as presented in Section 2) cannot be used in this example, since it was defined only for games where bilateral trades are mutually exclusive. In an article-in-progress, we extend the definition to more general cases, including this example by HM with compatible bilateral trades. The main generalization is the definition of *feasible events*, a subset of the power set of $\{(1,2),(1,3),(2,3)\}$. An event describes the set of bilateral agreements reached. For instance, the event $[(1,2),(1,3)]$ corresponds to 1 agreeing to trade with 2 *and* 1 also agreeing to trade with 3. If we represent with square brackets the events, in this paper the set of feasible events was $\{[(1,2)],[(1,3)],[(2,3)]\}$. In the example of HM, the set of events is $\{[(1,2),(1,3)],[(1,2)],[(1,3)],[(2,3)]\})$. That is, it contains the new event $[(1,2),(1,3)]$. In such a case, the extended $\epsilon-PSBN$ would still be a triple $\left\{u_i^{ij}(\epsilon), t_i^{ij}(\epsilon), p_e(\epsilon)\right\}$ except that $p_e(\epsilon)$ is a probability distribution over the set of feasible events (four). For our present purpose, let us keep the notation $p_{ij}(\epsilon)$ for the event $[(i,j)]$, and add $p(\epsilon)$ to denote the probability of event $[(1,2),(1,3)]$. Then, condition 1 in the definition of $\epsilon-PSBN$ remains unchanged. For condition 2 we need to introduce the concept of *mutually exclusive* pairs. Pairs (i,j) and (k,l) are mutually exclusive if there is no feasible event that contains both. In the previous sections, all three pairs mutually excluded each other. In the present example, however, $(1,2)$ and $(2,3)$ are mutually exclusive, as $(1,3)$ and $(2,3)$, but $(1,2)$ and $(1,3)$ are not. The relation is symmetric. Then condition 2 in the definition of an $\epsilon-PSBN$ implies that $t_i^{ij}(\epsilon)$ equals the expected payoff for player i in pairs that are mutually exclusive with (i,j), conditional on events that do not contain (i,j). In the present example, dropping the index ϵ for clarity,

$$t_i^{1i} = \frac{p_{23}u_i^{23}}{1-(p+p_{1i})},$$

for $i=2,3$. On the other hand, since $(1,3)$ and $(1,2)$ are not mutually exclusive, and 1 is not in the pair $(2,3)$, we have $t_1^{1i} = 0$. Also,

$$t_i^{23} = \frac{(p + p_{1i})u_i^{1i}}{1 - p_{23}},$$

for $i = 2, 3$.

Finally, condition 3, still dropping the index ϵ, should now include $p_{12} + p_{13} + p_{23} + p = 1$ and the probability of a *trade* between players i and j cannot be higher than $1 - \epsilon$ for all i, j.[24] But now, the probability of an event should not be larger than ϵ, unless all players included in the pairs of the event obtain in that event a higher payoff than they do in the rest.

With this definition of an $\epsilon - PSBN$, we can define the $PSBN$ and the $R-$solution for this more general case, just as in Section 2. It can be shown that if the two most efficient trades $\{(1,2), (1,3)\}$ can occur simultaneously and $V = v_{12} + v_{13}$, then the $R-$solution exists and is unique. In particular, $p = 1$, $U_1 = \frac{v_{12}+v_{13}}{2}$, and $U_i = \frac{v_{1i}}{2}$, for $i = 2, 3$, independently of the value of v_{23}. Hence, in HM's specific example, since $v_{23} = 0$, the $R-$solution and the Shapley value (the solution concept used in HM) offer the same prediction. Thus, under the $R-$solution, incentives to invest in this particular example do not change if we transfer property rights from the agent that takes investment decisions (the chef) to the indispensable agent (the tycoon). However, it is important to bear in mind that, like in the case of mutually exclusive bilateral trades, with more than one compatible bilateral trade, the $R-$solution and the Shapley value coincide only for very specific parameter values (in our restricted set up they only coincide if $v_{23} = 0$).

5.3 More than three players

In three-player games, if the grand coalition breaks down then only a two-player trade can be realized. The third player has to be in a one-player coalition. Thus, success in one bilateral coalition uniquely defines a partition of players. This introduces a natural hierarchy in the set of coalitions, and

[24] In particular, $p + p_{12}, p + p_{13}$, and p_{23} must all be lower or equal than $1 - \epsilon$.

it has allowed us to discuss our solution concept in terms of which bilateral coalition "succeeds". If $n > 3$ and the grand coalition fails, this hierarchy is absent even if only disjoint coalitions can form. The alternative to the grand coalition may be a $n-1-$player coalition, excluding the nth player, but it may also be $\frac{n}{2}$ disjoint two-player coalitions (in case n is an even number), or many other partitions. Specifying the disagreement points of a particular player in an arbitrary coalition can still be done, but it clearly involves a higher degree of complexity.

In order to gain a flavor of how to approach the analysis of $n-$player games, consider the four player example discussed by Bennett (1997), where only the following bilateral coalitions can generate a positive surplus: $v_{12} = 40, v_{13} = 34$, $v_{24} = 34, v_{34} = 20$. The value of all other coalitions is 0. We may consider this an example with two unit-demand buyers and two unit-supply sellers and also should define $V = v_{13} + v_{24} = 68$. Thus, each player can form a bilateral coalition with two other alternative players, and two simultaneous coalitions are feasible as long as they are disjoint. In line with the previous subsection, we can now define events, which correspond to feasible partitions of the set of players. In this case, there are the following events:[25] in event (a), which occurs with probability p_a, coalitions $\{1,3\}$ and $\{2,4\}$ form ; in event (b), which occurs with probability p_b, coalitions $\{1,2\}$ and $\{3,4\}$ form; and then there are events where only one two-player coalition forms, which we denote $(1,2), (1,3), (2,4), (3,4)$ and that occur with probabilities p_{12}, p_{13}, p_{24}, and p_{34} respectively. Note there are now coalitions that belong to two different events. For instance, $\{1,3\}$ belongs to event $(1,3)$ and to event (a). Let E be any set of events and denote by $p(E)$ the total probability of that set of events. Consider the following modification of condition 3 in the the definition of an $\epsilon - PSBN$.

$3'$) An event $e \in E$ has probability larger than $\epsilon p(E)$ only if every player

[25] Once again, we ignore the finest partition (no agreement at all) but considering it would not introduce any change of substance.

i, belonging to a coalition other than $\{i\}$ in event e, weakly prefers her payoff in that event to her payoff in any other event in E.

When E is the set of all events, we are back to our original condition. Thus, this is simply a strengthening of condition 3) in our definition. [26]

It can be shown (details available upon request) that the unique $PSBN$ includes $p_a = 1$, $u_1^{13} = u_1^{12} = u_2^{24} = u_2^{12} = 20$, $u_3^{13} = u_4^{24} = 14$, $u_3^{34} = u_4^{34} = 0$. There is no role for the grand coalition, and as a result the $R-$solution predicts the following payoffs: $U_1 = U_2 = 20$, $U_3 = U_4 = 14$. This result is analogous to the prediction of the $R-$solution for three-player games in Region 2. In other words, the efficient set of coalitions forms, and players 1 and 2 are able to jointly obtain the exact value of their outside opportunity. In contrast, Bennett (1997) predicts a higher payoff for players 1 and 2 (she predicts $U_1 = U_2 = 22$). Such a division of surplus cannot be justified on the grounds of purported consistent beliefs. Players 1 and 2 cannot simultaneously claim a payoff of 22 in their negotiation with players 3 and 4 respectively, since the threat of breaking the negotiations and forming the coalition $(1, 2)$ would necessarily involve losses for both players.

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[26] It can be checked that the $\epsilon - PSBN$ specified in the existence proof still satisfies this more stringent condition.

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7 Appendix

7.1 Proof of Proposition 1:

First we propose an $\epsilon - PSBN$ for the game (N, v) for ϵ sufficiently small. This will show existence. To save in notation, we will dispose of the (ϵ) index of the solution, and instead specify if we refer to the limit.

1) Let $\frac{1}{2}v_{12} \geq v_{13}$.

1.a) If $\frac{1}{2}v_{12} > v_{13}$ (so that $v_{12} \geq v_{13} + v_{23}$ is also satisfied), consider $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$, and $u_i^{ij} = 0$ for all other values of i, j. Also, let $p_{12} = 1-\epsilon$, $p_{13} = p_{23} = \frac{\epsilon}{2}$. Finally, let $t_1^{12} = t_2^{12} = t_3^{i3} = 0$ and $t_i^{i3} = \frac{1-\epsilon}{2-\epsilon}v_{12}$ for $i = 1, 2$. Note that $\lim_{\epsilon \to 0} \frac{1-\epsilon}{2-\epsilon}v_{12} = \frac{1}{2}v_{12} > v_{13} \geq v_{13}$. Thus, for ϵ sufficiently small, this satisfies the definition of an $\epsilon - PSBN$.

1.b) If $\frac{1}{2}v_{12} = v_{13} > v_{23}$, consider $u_1^{12} = u_1^{12} = \frac{1}{2}v_{12}$ $(= v_{13})$, and $u_2^{23} = u_3^{23} = 0$. Also, let $p_{12} = 1 - \epsilon$ and $p_{13} = 0$, $p_{23} = \epsilon$. Then, $t_1^{12} = t_3^{23} = t_2^{12} = t_3^{13} = 0$ and $t_2^{23} = \frac{1}{2}v_{12} > v_{23}$. To complete the definition of an $\epsilon - PSBN$ we need only $t_1^{13} = (1 - \epsilon)\frac{1}{2}v_{12}$, $u_1^{13} = \frac{1}{2}(v_{13} + t_1^{13} - t_3^{13}) = (1 - \frac{\epsilon}{2})v_{13}$ and $u_3^{13} = \frac{\epsilon}{2}v_{13}$.

1.c) If $v_{13} = v_{23} = \frac{1}{2}v_{12}$, consider $u_1^{12} = u_2^{12} = \frac{1}{2}v_{12}$ $(= v_{i3},\ i = 1, 2)$, $p_{12} = 1 - \epsilon$ and $p_{13} = p_{23} = \frac{\epsilon}{2}$. Then $t_i^{i3} = \frac{(1-\epsilon)v_{12}}{2-\epsilon} < v_{i3}$, $i = 1, 2$. Also, consider $u_3^{13} = u_3^{23} = A > 0$. Thus, t_3^{i3}, $i = 1, 2$, will have to satisfy:

$$t_3^{i3} = \frac{\epsilon A}{2 - \epsilon}, \text{ and}$$

$$A = \frac{1}{2}\left(v_{i3} - \frac{(1-\epsilon)v_{12}}{2-\epsilon} + \frac{\epsilon A}{2-\epsilon}\right),$$

and solving for A given that $\frac{1}{2}v_{12} = v_{i3}$, we obtain

$$A = \frac{\epsilon v_{i3}}{4-3\epsilon},$$

which is smaller than v_{i3} for small ϵ. Note that for ϵ small $t_3^{i3} + t_i^{i3} < v_{i3}$, $i = 1, 2$. Also, note that given these values for u_i^{i3}, we should define $t_1^{12} = t_2^{12} = \frac{\frac{\epsilon}{2}(v_{i3}-A)}{\epsilon} = \frac{(v_{12}-2A)}{4}$, and $t_1^{12} + t_2^{12} < v_{12}$. This satisfies the definition of an $\epsilon - PSBN$.

2) If $v_{13} > \frac{1}{2}v_{12}$ and $v_{12} > v_{13}+v_{23}$, then consider $u_1^{13} = v_{13}$, $u_1^{12} = v_{13}+a$, and $u_3^{13} = u_2^{23} = u_3^{23} = 0$, where $a > 0$ is to be obtained later. Thus, $u_2^{12} = v_{12} - u_1^{12} > u_2^{23}$. Consequently, let $p_{23} = \epsilon$. Then $p_{12} = 1 - \epsilon - p_{13}$. Finally, $u_2^{23} = u_3^{23} = 0$ implies that $t_2^{12} = t_3^{13} = 0$, and we can then check that

$$t_1^{12} + t_2^{12} = \frac{p_{13}}{\epsilon + p_{13}} v_{13} < v_{12},$$

whereas

$$t_1^{13} + t_3^{13} = \frac{p_{12}(v_{13}+a)}{1-p_{13}},$$

which is smaller or equal than v_{13} if

$$a \leq v_{13}\frac{\epsilon}{p_{12}}. \tag{3}$$

We will propose a so that this is satisfied. In that case, a should also satisfy

$$v_{13} = \frac{1}{2}(v_{13} + \frac{(1-\epsilon-p_{13})(v_{13}+a)}{1-p_{13}}) = \frac{1}{2}(v_{12} + \frac{p_{13}v_{13}}{\epsilon+p_{13}}) - a. \tag{4}$$

We can solve the first equation for a and get $a = v_{13}\frac{\epsilon}{p_{12}}$. Thus, the condition (3) is satisfied by a solution to the above system. Note that in such a solution a converges to 0. This is a system with two unknowns and two variables. (For positive ϵ, indeed $a > 0$.) Also,

$$\frac{2\epsilon - p_{13}}{\epsilon + p_{13}} = \frac{v_{12}-2a}{v_{13}},$$

so that $p_{13} < \epsilon$, so that condition 3 is satisfied. Is there a valid solution to (4)? Substituting (3) with equality in the second equation of (4) we obtain two solutions for p_{13}, and one of them converges to zero:

$$p_{13} = \frac{1}{2}\left(1 - \sqrt{1 - 6\frac{\epsilon v_{13}}{v_{12}-v_{13}} + (\frac{\epsilon v_{13}}{v_{12}-v_{13}})^2 - \epsilon\frac{(2v_{12}-v_{13})}{v_{12}-v_{13}}}\right).$$

We can also check that for small but positive ϵ this expression is positive. Indeed, $1 > \sqrt{1 - 6\frac{\epsilon v_{13}}{v_{12}-v_{13}} + (\frac{\epsilon v_{13}}{v_{12}-v_{13}})^2 - \epsilon\frac{(2v_{12}-v_{13})}{v_{12}-v_{13}}}$, since $1 >$

$\left(\sqrt{1-6\frac{\epsilon v_{13}}{v_{12}-v_{13}}+(\frac{\epsilon v_{13}}{v_{12}-v_{13}})^2}-\epsilon\frac{(2v_{12}-v_{13})}{v_{12}-v_{13}}\right)^2$ given that the right hand side is smaller than

$$1-6\frac{\epsilon v_{13}}{v_{12}-v_{13}}+(\frac{\epsilon v_{13}}{v_{12}-v_{13}})^2+\left(\epsilon\frac{(2v_{12}-v_{13})}{v_{12}-v_{13}}\right)^2,$$

and the two last terms approach zero at the speed of ϵ^2, whereas the second approaches zero at the slower speed of ϵ.

3) If $v_{12} \leq v_{13}+v_{23}$, and $v_{23}>0$, then propose $u_i^{ij}=u_i^{ik}=u_i$, for all i,j,k, with $u_i+u_j=v_{ij}$ for all i,j. This is a system of three linear (independent) equations with solution $u_i=\frac{v_{ij}+v_{ik}-v_{jk}}{2}$. These values are strictly positive. Also, $t_i^{ij}=\frac{p_{ik}u_i}{1-p_{ij}}$. Finally, p should satisfy

$$u_i=\frac{1}{2}(v_{ij}+\frac{p_{ik}u_i}{p_{ik}+p_{jk}}-\frac{p_{jk}u_j}{p_{ik}+p_{jk}})$$

for all i,j,k. Taking into account $u_i+u_j=v_{ij}$, these equations can be written as

$$\begin{aligned}
-p_{13}u_2+p_{23}u_1 &= 0,\\
-p_{12}u_3+p_{13}u_2 &= 0,\\
-p_{12}u_3+p_{23}u_1 &= 0.
\end{aligned}$$

Note that the third equation is simply the sum of the previous two. That is, there are only two linearly independent equations. Thus, two of these equations plus $p_{13}+p_{23}+p_{23}=1$, forming a linear system with a unique solution. The solution is a probability distribution, where all three variables take strictly positive values. Indeed, the first two equations can be written as $\frac{p_{13}}{u_1}=\frac{p_{23}}{u_2}$ and $\frac{p_{12}}{u_2}=\frac{p_{13}}{u_3}$, so that all solution vectors to these two equations have either all positive or all negative components. And no solution with all negative components satisfies the equation $p_{13}+p_{23}+p_{23}=1$. Finally, note that $t_j^{ij}+t_i^{ij}=\frac{p_{jk}u_j}{p_{jk}+p_{ik}}-\frac{p_{ik}u_i}{p_{jk}+p_{ik}}$, so that since both $u_j,u_i<v_{ij}$, indeed $t_j^{ij}+t_i^{ij}<v_{ij}$. Thus, we have a $\epsilon-PSBN$ for ϵ sufficiently small ($\epsilon<p_{23}$ obtained above).

If $v_{23}=0$ and so $v_{12}=v_{13}$ any p_{12}, $p_{13}<1-\epsilon$ so that $p_{12}+p_{13}=1$, with $u_2=u_3=0$, and $u_1=v_{12}$, and correspondingly $t_1^{12}=t_1^{13}=v_{12}$ and $t_i^{ij}=0$ otherwise, form a $\epsilon-PSBN$.

This concludes the proof of existence. Next, we can simply check that if we select the $\epsilon-PSBN$ that we have just characterized for each possible values of v_{ij} for all ij, then the $\lim_{\epsilon\rightarrow 0}\{u(\epsilon),t(\epsilon),p(\epsilon)\}$ is as stated in the Proposition. (There is an exception, when $v_{23}=0$ and $v_{12}=v_{13}$, in which case the limit is not unique in probabilities, yet is in all outcome-relevant values.) Thus, we only need showing that there is no other triple $\{u,t,p\}$ that is the limit of a sequence of $\epsilon-PSBN$ as ϵ approaches 0. First, however, we will prove a handy result.

Lemma 2 *In a $\epsilon - PSBN$, cycles cannot occur. That is, it cannot be that $u_i^{ij} \geq u_i^{ik}$; $u_j^{jk} \geq u_j^{ij}$; $u_k^{ik} \geq u_k^{jk}$ for some values of i, j, k. Moreover, $u_i^{ij} = u_i^{ik}$; $u_j^{jk} = u_j^{ij}$; $u_k^{ik} = u_k^{jk}$ can only occur if $v_{12} \leq v_{13} + v_{23}$.*

Proof of Lemma: Assume such a cycle with at least one strict inequality. First, assume that $t_i^{ij} + t_j^{ij} \leq v_{ij}$ for all ij. That requires $u_i^{ij} = \frac{1}{2}\left(v_{ij} + t_i^{ij} - t_j^{ij}\right)$ for all i, j, k. Substituting for $v_{ij} = u_i^{ij} + u_j^{ij}$, and also substituting for

$$t_i^{ij} = \frac{p_{ik}}{1 - p_{ij}} u_i^{ik} \tag{5}$$

we can write this expression as

$$(u_i^{ij} - u_j^{ij})(1 - p_{ij}) = p_{ik} u_i^{ik} - p_{jk} u_j^{jk} \tag{6}$$

Adding these three equations, for all three pairs, this implies

$$(u_i^{ij} - u_j^{ij}) + (u_k^{ik} - u_i^{ik}) + (u_j^{jk} - u_k^{jk}) = 0,$$

that is, $u_i^{ij} + u_k^{ik} + u_j^{jk} = u_j^{ij} + u_i^{ik} + u_k^{jk}$; this violates the inequalities defining the cycle if there is one that is strict.

Second, assume that $t_i^{ij} + t_j^{ij} > v_{ij}$ for some ij, but $t_i^{ik} + t_k^{ik} \leq v_{ik}$, and $t_j^{jk} + t_k^{jk} \leq v_{jk}$. This and the inequalities defining the cycle imply $u_i^{ij} = u_j^{ij} = u_i^{ik} = 0$, so that also $u_k^{ik} = v_{ik}$. Thus, equation (6) for the pair jk becomes

$$(u_j^{jk} - u_k^{jk})(1 - p_{jk}) = -p_{ik} v_{ik}.$$

Since $p_{jk} < 1$, that implies $u_k^{jk} \geq u_j^{jk}$. Note, however, that $t_j^{jk} = 0$, since $u_j^{ij} = 0$, so that $u_j^{jk} \geq u_k^{jk}$. These two inequalities then imply both $u_j^{jk} = u_k^{jk} = \frac{v_{jk}}{2}$, and $p_{ik} = 0$. Since the inequalities that define the cycle include $u_k^{ik} \geq u_k^{jk}$, then we must have $v_{ik} \geq \frac{v_{jk}}{2}$. But substituting for $u_k^{jk} = \frac{v_{jk}}{2}$ and $p_{ik} = 0$ in (5) corresponding to t_k^{ik}, we also have that $t_k^{ik} = p_{jk}\frac{v_{jk}}{2} < \frac{v_{jk}}{2} \leq v_{ik}$. This contradicts that $u_k^{ik} = v_{ik}$.

Third, assume that $t_i^{ij} + t_j^{ij} > v_{ij}$ and $t_i^{ik} + t_k^{ik} > v_{ik}$ for some ij and ik but $t_j^{jk} + t_k^{jk} \leq v_{jk}$. That implies that $u_i^{ij} = u_j^{ij} = u_i^{ik} = u_k^{ik} = 0$, which implies that $t_j^{jk} = t_k^{jk} = 0$, so that $u_k^{jk} = \frac{v_{jk}}{2} > u_k^{ik}$, which contradicts the inequalities that define the cycle.

Thus, the only cycle that may exist is $u_i^{ij} = u_i^{ik}$; $u_j^{jk} = u_j^{ij}$; $u_k^{ik} = u_k^{jk}$, with $t_i^{ij} + t_j^{ij} \leq v_{ij}$ for all ij. But the system $u_i + u_j = v_{ij}$, for all ij has a valid solution only in Region 3, and coincides with the one found above. QED

Thus, an $\epsilon - PSBN$ must satisfy:

$$u_i^{ij} \geq u_i^{ik}; u_j^{jk} \leq u_j^{ij}; u_k^{ik} \leq u_k^{jk}, \tag{7}$$

and except for the $\epsilon - PSBN$ in 3) above, at least two inequalities must be strict. Also, given part three of the definition of $\epsilon - PSBN$, $p_{ik} < \epsilon$ unless $u_i^{ij} = u_i^{ik}$ and $u_k^{ik} = u_k^{jk}$. Thus, in any but the $\epsilon - PSBN$ in 3) above, $p_{ik} < \epsilon$. Thus, in a sequence that converges as $\epsilon \to 0$, we must have $\lim_{\epsilon\to 0} p_{ik} = 0$.

Consider such a sequence of $\epsilon - PSBN$'s so that $\lim_{\epsilon\to 0} p_{ij} > 0$ and $\lim_{\epsilon\to 0} p_{jk} > 0$. From (7) and part three of the definition of $\epsilon - PSBN$, that implies for ϵ small $u_j^{ij} = u_j^{jk}$. Thus, since at least two inequalities need to be strict, $u_i^{ij} > u_i^{ik}$ and $u_k^{ik} < u_k^{jk}$. These last inequalities imply that $u_k^{jk} + u_j^{jk} = v_{jk}$ and $u_i^{ij} + u_j^{ij} = v_{ij}$. Also, as $\epsilon \to 0$, $\frac{p_{jk}}{p_{jk}+p_{ik}} \to 1$, and $\frac{p_{ij}}{p_{ij}+p_{ik}} \to 1$, so that applying part one of the definition of a $\epsilon - PSBN$,

$$u_j^{ij} = u_j^{jk} \to u_j = \frac{1}{2}(v_{ij} + u_j) = \frac{1}{2}(v_{jk} + u_j).$$

This cannot occur unless $v_{ij} = v_{jk}$. In the latter case, $u_j = v_{ij} = v_{jk}$, which implies both u_i^{ij} and u_k^{jk} converge to 0, and so $t_i^{ik} + t_k^{ik}$ converges to 0, in which case u_i^{ik} converges to $\frac{v_{ik}}{2} > u_i^{ij}$ for ϵ small when $v_{ik} > 0$. This is a contradiction unless $v_{ik} = 0$. But if $v_{ij} = v_{jk}$, $v_{ik} = 0$, the limit of such a sequence coincides with the $\epsilon - PSBN$ in 3) above.

Thus, it must be the case that these two conditions hold: (i) $\lim_{\epsilon\to 0} p_{ik} = 0$, and (ii) either $\lim_{\epsilon\to 0} p_{jk} = 0$ or $\lim_{\epsilon\to 0} p_{ij} = 0$. But if $\lim_{\epsilon\to 0} p_{ij} = 0$ then $\lim_{\epsilon\to 0} p_{jk} > 0$, and this contradicts part 3 of the definition of an $\epsilon - PSBN$ since $u_i^{ij} \geq u_i^{ik}$ and $u_j^{ij} \geq u_j^{jk}$ with at least one inequality. Thus, assume that $\lim_{\epsilon\to 0} p_{ik} = \lim_{\epsilon\to 0} p_{jk} = 0$. We consider two possible cases:

1) Assume that, in the limit, $t_j^{jk} + t_k^{jk} > v_{jk}$. Other than in the trivial case where $v_{jk} = 0$, this requires that for ϵ small the inequality is also satisfied, so that for ϵ small $u_j^{jk} = u_k^{jk} = 0 = t_j^{ij}$. Thus, from (7), we must also have that $u_k^{ik} = 0$. Since only one inequality in (7) may be non strict, and $u_k^{jk} = u_k^{ik}$, we must have $u_i^{ij} > u_i^{ik}$, and since $u_j^{jk} = 0$, we must also have that $u_j^{ij} > u_j^{jk}$. These two inequalities imply that $u_i^{ij} + u_j^{ij} = v_{ij}$. Since $t_j^{ij} = 0$, we must then have that $u_i^{ij} \geq \frac{v_{ij}}{2}$. Thus:

1.a) If $\frac{v_{ij}}{2} > v_{ik}$, since t_i^{ik} converges to $u_i^{ij} \geq \frac{v_{ij}}{2}$, then for ϵ small we must also have that $t_i^{ik} + t_k^{ik} > v_{ik}$, so that $u_i^{ik} = 0$, and then $t_j^{ij} = t_i^{ij} = 0$, and then $u_j^{ij} = u_i^{ij} = \frac{v_{ij}}{2}$. Note that $t_j^{jk} \leq u_j^{ij}$ and $t_k^{jk} = 0$. Thus, for $t_j^{jk} + t_k^{jk} > v_{jk}$, it must be that $\frac{v_{ij}}{2} > v_{jk}$. This requires that $ij = 1,2$ and also that we are in Region 1. Thus, the limit of such a sequence is the one stated in the Proposition.

1.b) If $\frac{v_{ij}}{2} \leq v_{ik}$, as before, and if $t_i^{ik} + t_k^{ik} > v_{ik}$, then $u_j^{ij} = \frac{v_{ij}}{2}$, and since $t_k^{ik} = 0$, this would imply that $\frac{v_{ij}}{2} \geq t_i^{ik} > v_{ik}$, which is a contradiction. Thus, we must have $t_i^{ik} + t_k^{ik} \leq v_{ik}$. Thus, since $u_i^{ik} = 0$, we must have $u_k^{ik} = v_{ik}$. Since $u_k^{jk} = 0$, this contradicts the inequality $u_k^{ik} \leq u_k^{jk}$ in (7), unless $v_{ik} = 0$. In the latter case, since $\frac{v_{ij}}{2} \leq v_{ik}$, $v_{ij} = 0$ and we have a contradiction with $t_j^{jk} + t_k^{jk} = 0 > v_{jk}$.

2) Assume that $t_j^{jk} + t_k^{jk} \leq v_{jk}$ in all the terms of the sequence as ϵ converges to 0.

2.a) If $t_i^{ij} + t_j^{ij} \leq v_{ij}$, then

$$u_j^{jk} \geq t_j^{jk} = \frac{p_{ij}}{p_{ij} + p_{ik}} u_j^{ij},$$

where the right hand side converges to u_j^{ij}. From, (7), $u_j^{ij} \geq u_j^{jk}$. Thus, the limit of any such sequence should satisfy $\lim_{\epsilon\to 0} u_j^{ij} = \lim_{\epsilon\to 0} u_j^{jk} = \lim_{\epsilon\to 0} t_j^{jk}$. That implies that $\lim_{\epsilon\to 0} u_k^{jk} = 0$, and requires that $v_{ij} \geq v_{jk}$, and $\lim_{\epsilon\to 0} u_i^{ij} = v_{ij} - v_{jk}$. Since $u_k^{jk} \geq u_k^{ik}$, then we also have $\lim_{\epsilon\to 0} u_k^{ik} = 0$. But if $\lim_{\epsilon\to 0} u_k^{jk} = 0$, then $\lim_{\epsilon\to 0} t_k^{ik} = 0$, whereas $t_i^{ik} \leq u_i^{ik}$. Thus, if $v_{ik} > v_{ij} - v_{jk}$, then $\lim_{\epsilon\to 0} t_i^{ik} + t_k^{ik} < v_{ik}$ and then $\lim_{\epsilon\to 0} u_k^{ik} > \frac{v_{ik} - (v_{ij} - v_{jk})}{2} > 0$, which is a contradiction. Therefore, $v_{ik} \leq v_{ij} - v_{jk}$. Since $\lim_{\epsilon\to 0} t_i^{ik} = \lim_{\epsilon\to 0} u_i^{ij} = v_{ij} - v_{jk}$, then $\lim_{\epsilon\to 0} u_k^{ik} = 0 = \lim_{\epsilon\to 0} u_k^{jk}$. Thus, $\lim_{\epsilon\to 0} u_i^{ik} > 0$, only if $\lim_{\epsilon\to 0} u_i^{ik} = v_{ij} - v_{jk}$. In this case, we would have $u_i = \lim_{\epsilon\to 0} u_i^{ij} = \lim_{\epsilon\to 0} u_i^{ik}$, $u_k = \lim_{\epsilon\to 0} u_k^{ik} = \lim_{\epsilon\to 0} u_k^{jk}$, and $u_j = \lim_{\epsilon\to 0} u_j^{ij} = \lim_{\epsilon\to 0} u_j^{jk}$. This equation, together with $u_i + u_k = v_{ik}$, $u_i + u_j = v_{ij}$ $u_j + u_k = v_{jk}$ has a solution only in Region 3 ($v_{ik} = v_{ij} - v_{jk}$). Thus, if $v_{ik} < v_{ij} - v_{jk}$, $u_i^{ik} = 0$ for ϵ small, so that $t_i^{ik} = 0$, so that $u_j^{ij} \geq \frac{v_{ij}}{2}$, and then $v_{jk} \geq \frac{v_{ij}}{2}$. This is Region 2, and the limit coincides with the one stated in the Proposition.

2.b) If $t_i^{ij} + t_j^{ij} > v_{ij}$, then u_i^{ij} $(= u_j^{ij}) = 0$, so that $t_i^{ik} = 0$, and since from (7) $u_i^{ij} \geq u_i^{ik}$, then $u_i^{ik} = 0$. On the other hand, t_k^{ik} approaches 0 as $\epsilon \to 0$, and then $t_i^{ik} + t_k^{ik}$ approaches 0, which contradicts $u_i^{ik} = 0$ unless $v_{ik} = 0$. Moreover, in this latter case $t_j^{jk} = t_k^{jk} = 0$, so that $u_j^{jk} = u_k^{jk} = \frac{v_{jk}}{2}$, so that $u_j^{jk} > u_j^{ij}$, which contradicts (7).

7.2 Proof of Proposition 2

We first show existence of a stationary, SPE for δ close to 1. Note that in equilibrium any offer/demand θ_i^j that is accepted with positive probability needs to satisfy $\theta_i^j = v_{ij} - \delta x_j$, where x_j is the equilibrium payoff of player j, $j = 1, 2, 3$.

Assume we are in Region 1, where $v_{12} \geq 2v_{13}$. Then it is easy to check that $\mu_2^1 = \mu_1^2 = \lambda_2^1 = \lambda_1^2 = 1$, with λ_3^1 and μ_3^1 arbitrary, $\theta_2^1 = \theta_1^2 = \frac{2-\delta}{2} v_{12}$, $\rho_2^1 = \rho_1^2 = 1$, (and $\rho_i^j = 0$ for any $(i, j) \neq (1, 2)$ and, for instance, $\theta_i^3 > v_{i3}$ and $\theta_3^i = 0$ for $i = 1, 2$), is an equilibrium of the game.

Assume we are in Region 2 where $v_{12} < 2v_{13}$ and $v_{12} \geq v_{13} + v_{23}$. We will propose an equilibrium where agreement will occur in period 1. We propose $\mu_2^1 = \mu_3^1 = \mu_1^2 = \lambda_2^1 = \lambda_3^1 = 1$. Finally, $\lambda_1^3 = \frac{3}{2}\varpi$ for some (small) value of ϖ to be determined. That means that players 1 and 2 will meet with probability $1-\frac{2}{3}\lambda_1^3 = 1-\varpi$. For that to be an equilibrium with agreement in period 1, we require player 1 to be indifferent between meeting player 2 and meeting player 3, when she receives a first invitation to meet. That is,

$$\frac{1}{2}(v_{13}-\delta x_3+\delta x_1) = \frac{1}{2}(v_{12}-\delta x_2+\delta x_1).$$

Indeed, the left hand side is the (equilibrium) expected payoff for player 1, conditional on meeting player 3, and the right hand side is her payoff, conditional on meeting player 2. This is so since equilibrium offers must satisfy $\theta_1^i = v_{1i}-\delta x_i$ and $\theta_i^1 = v_{1i}-\delta x_1$ for $i = 2,3$. Thus, in such equilibrium the following must be satisfied:

$$v_{13}-\delta x_3 = v_{12}-\delta x_2. \tag{8}$$

Also, we consider an equilibrium where all offers are made and accepted when player 1 meets another player, but only in this case. Equilibrium payoffs in that case must satisfy

$$\begin{aligned} x_3 &= \frac{\varpi}{2}(v_{13}-\delta x_1+\delta x_3), \\ x_2 &= \frac{1-\varpi}{2}(v_{12}-\delta x_1+\delta x_2), \\ x_1 &= \frac{1}{2}(v_{12}-\delta x_2+\delta x_1). \end{aligned}$$

If $v_{12} > v_{13}$ these four equations have a (positive) solution:

$$\begin{aligned} x_1 &= \frac{1}{2(4-3\delta)}((3-2\delta)(v_{13}+v_{12})-H), \\ x_2 &= \frac{1}{2\delta(4-3\delta)}((2-\delta)(H+(v_{12}-v_{13})(3-2\delta))-4(1-\delta)^2 v_{12}), \\ x_3 &= \frac{1}{2\delta(4-3\delta)}(2-\delta)(H-(v_{12}-v_{13})(3-2\delta))-4(1-\delta)^2 v_{13}). \\ \varpi &= \frac{v_{12}+v_{13}(1-2\delta)-H}{2\delta(v_{12}-v_{13})}, \end{aligned}$$

where

$$H = \left[((v_{12}-v_{13})(3-2\delta))^2 + 4v_{12}v_{13}(1-\delta)^2\right]^{\frac{1}{2}}.$$

That is,

$$0 < \varpi < \frac{1-\delta}{\delta}\frac{2v_{13}-v_{12}}{v_{12}-v_{13}}.$$

As δ approaches 1, the upper bound on ϖ tends to 0, and x_1 approaches v_{13}, x_2 approaches $v_{12}-v_{13}$ and x_3 approaches 0. Note that $H > (v_{12}-v_{13})(3-2\delta)$, so that $x_1 < \frac{3-2\delta}{4-3\delta}v_{13} < v_{13}$ and $x_2 > v_{12}-v_{13} \geq v_{23}$. Thus, player 2 can never be tempted to negotiate with player 3. Since $v_{12} < 2v_{13}$ then for δ sufficiently high, $x_1 > x_2$ and player 1 cannot be tempted either to negotiate with player 3. Thus, we have an equilibrium with only completing $\theta_1^j = v_{1j} - \delta x_j$ and $\theta_j^1 = v_{1j} - \delta x_1$, $\theta_j^i > v_{ji}$ otherwise, $\rho_j^1 = \rho_1^j = 1$ and $\rho_3^2 = \rho_2^3 = 0$, for these values of x_j obtained. Finally, if $v_{12} = v_{13}$ then $v_{23} = 0$, and the solution to the above system of equations includes $\varpi = \frac{1}{2}, x_1 = \frac{v_{12}(2-\delta)}{4-3\delta} < v_{12}$, and $x_2 = x_3 = \frac{v_{12}\left(4-5\delta+\delta^2\right)}{(4-3\delta)(4-\delta)} > 0$. As δ approaches 1 x_1 approaches v_{12}, and x_2 and x_3 approach 0. Moreover, we can complete the description of equilibrium as above.

Now, assume that we are in Region 3, where $v_{12} < v_{13} + v_{23}$. Then, consider an equilibrium where all players are indifferent about meeting either of the other two players, if they receive the first invitation to meet, and all meetings end in agreement. That is, (8) still holds, but this time for all pairs of players. These are three equations, but only two of them are linearly independent. Let us define

$$p_{ij} = \frac{1}{3}(\mu_j^i + \mu_k^i)\lambda_i^j + \frac{1}{3}(\mu_i^j + \mu_k^j)\lambda_j^i. \tag{9}$$

(Note that $\mu_k^i + \mu_k^j = 1$, so that $p_{ij} \leq 1$.) Then, the payoffs should satisfy:

$$x_i = p_{ij}(v_{ij} - \delta x_j + \delta x_i) + p_{ik}(v_{ik} - \delta x_k + \delta x_i).$$

Moreover,

$$p_{12} + p_{13} + p_{23} = 1. \tag{10}$$

Thus, we have a system of six equations with six unknowns, the three values p and the three values x. Let us denote by $A = \frac{v_{12}+v_{13}-v_{23}}{2}$, $B = \frac{v_{12}+v_{23}-v_{13}}{2}$ and $C = \frac{v_{13}+v_{23}-v_{12}}{2}$. The system of equations has a solution:

$$\begin{aligned}
x_1 &= \frac{A(AB + AC - (2-\delta)BC)}{(AB + AC + BC)}, \qquad (11)\\
x_2 &= \frac{B(AB + BC - (2-\delta)AC)}{\delta\,(AB + AC + BC)},\\
x_3 &= \frac{C(AC + BC - (2-\delta)AB)}{\delta\,(AB + AC + BC)}.\\
p_{12} &= \frac{(2-\delta)AB - (AC + BC)\,(1-\delta)}{\delta\,(AB + AC + BC)},\\
p_{13} &= \frac{(2-\delta)AC - (AB + BC)\,(1-\delta)}{\delta\,(AB + AC + BC)}.
\end{aligned}$$

As δ approaches 1, these values are a valid solution, with all p positive. We should set $\rho_i^j = 1$ and $\theta_i^j = v_{ij} - \delta x_j$, for all i, j. Also, we can recover values μ_i^j and λ_i^j from the values p_{ij}. That is, solutions to (9) with these values of the p's that also satisfy $\lambda_i^j + \lambda_i^k = 1$ and $\mu_i^k + \mu_i^j = 1$, and similarly for all players. We propose an equilibrium where $\mu_2^1 = 1$, and the other two players are indifferent as to whom to make the first invitation to meet when they are chosen by Nature to make this invitation. That requires that $\lambda_2^3 = \lambda_1^3$ and $\lambda_2^1 = \lambda_3^1$. Since $\lambda_2^3 + \lambda_2^1 = 1$, this amounts to requiring that $\lambda_1^3 + \lambda_3^1 = 1$. Substituting the solution for the values of p_{ij} in (9), (letting $\lambda_i^j = 1 - \lambda_i^k$ for all i and $\mu_i^k = 1 - \mu_i^j$ for $i = 1$ and 2) together with this latter equation, we obtain a system with four equations with four unknowns: μ_1^2, μ_3^1, λ_3^1 $(= \lambda_2^1)$ and λ_1^3. One of these equations is linearly dependent, so we can set, for instance, $\mu_1^2 = 1$, and solve the system to obtain:

$$\begin{aligned}
\mu_3^1 &= \frac{2p_{13} - p_{23}}{p_{13} + p_{23}} = \frac{C(2A - B) + 3(1-\delta)\left(AC - BC - AB\right)}{C(A+B) - 2(1-\delta)AB}, \\
\lambda_3^1 &= p_{12}, \\
\lambda_1^3 &= p_{13} + p_{23}.
\end{aligned}$$

Recall that $A \geq B$ and, in Region 3, $A < 2B$, and then for δ large enough, μ_3^1 approaches $\frac{2A-B}{A+B} \in (0,1)$. This completes the proof of existence.

In order to prove the convergence result, we need to show that as $\delta \longrightarrow 1$ all equilibria result in outcomes $\mathbf{x}$ arbitrarily close to the ones we constructed in the first part of this proof. Before proceeding to the proof, we start with a few helpful results:

Result 1: In a SPE a cycle, where i (weakly) prefers meeting j, who (weakly) prefers meeting k, who (weakly) prefers meeting i, can only occur when all players are indifferent between meeting the two possible partners.

Proof: Assume $v_{lm} \geq \delta x_l + \delta x_m$ for all $l, m = 1, 2, 3$. When it is the opponent's turn to make an offer, i receives the same offer from both, j and k: to keep δx_i. Moreover, this is precisely i's payoff if the partner, j or k, decides to make no offer. Thus, preference could only come from i being able to make a (weakly) higher, acceptable demand to j than to k: $v_{ij} - \delta x_j \geq v_{ik} - \delta x_k$. Thus, the cycle means $v_{ij} - \delta x_j \geq v_{ik} - \delta x_k$, $v_{jk} - \delta x_k \geq v_{ij} - \delta x_i$, and $v_{ik} - \delta x_i \geq v_{jk} - \delta x_j$, which is only possible if all these inequalities hold with equality. Assume now that $v_{ik} < \delta x_i + \delta x_k$, but $v_{ij} \geq \delta x_j + \delta x_j$ and $v_{jk} \geq \delta x_j + \delta x_k$. In this case, and by a similar argument the cycle implies that $v_{ij} - \delta x_j \geq \delta x_i$, $v_{jk} - \delta x_k \geq v_{ij} - \delta x_i$, and $\delta x_k \geq v_{jk} - \delta x_j$. The first and third inequalities can be rewritten as $v_{ij} - \delta x_i \geq \delta x_j$ and $\delta x_j \geq v_{jk} - \delta x_k$, so that again this can be possible only with all three expressions holding with equality. Assume now that $v_{ik} < \delta x_i + \delta x_k$ and $v_{jk} < \delta x_j + \delta x_k$ and $v_{ij} \geq \delta x_j + \delta x_j$. Thus, a cycle implies that $v_{ij} - \delta x_j \geq \delta x_i$, $\delta x_j \geq v_{ij} - \delta x_i$, and $\delta x_k \geq \delta x_k$, which again implies that all three expressions hold with equality.

Result 2: In a SPE, if $\delta x_i + \delta x_j < v_{ij}$ then $\rho_i^j = \rho_j^i = 1$.

Proof: The proof is trivial.

Result 3: In a SPE where $\delta x_i + \delta x_j \geq v_{ij}$, if $\lambda_i^j > 0$ then $x_i = 0$.

Proof: Suppose that $\lambda_i^j > 0$, and $x_i > 0$. Then since $\delta x_i + \delta x_j \geq v_{ij}$, we have that the expected payoffs at step (4), if players i and j meet, are δx_i and δx_j, respectively. Indeed, this is the payoff if no offer is made or if offers are rejected, and also this is the highest offer each would get or the highest demand each can make and (perhaps) be accepted. Since $\delta x_i < x_i$ when $x_i > 0$, it must be that the expected payoff for player i at step (4) when meeting player k is higher than δx_i, and so $\lambda_i^j = 0$ is a profitable deviation for player i at step (3).

We now proceed to prove the convergence result.

1) First consider an equilibrium where $\delta x_i + \delta x_j < v_{ij}$ for all i, j, so that in that equilibrium $\rho_i^j = 1$, for all i, j (and all offers are accepted with probability one). In such equilibrium, when player i proposes to player j in step 5, the proposal must be:

$$\theta_i^j = v_{ij} - \delta x_j, \tag{12}$$

and therefore, player j expects to keep the same payment whether she is proposed to by i or by k: δx_j. Also,

1.1) Suppose that in such equilibrium all players, when they are chosen to receive the first invitation to meet in step (2), are indifferent about accepting or instead selecting to meet the other player. For player j to be indifferent to meet i or k, we need $\theta_j^i = \theta_j^k \equiv \theta_j$. Therefore, equation (12) gives us conditions for such equilibrium

$$v_{12} - \theta_1 = v_{23} - \theta_3; v_{13} - \theta_1 = v_{23} - \theta_2, \tag{13}$$

which already imply $v_{12} - \theta_2 = v_{13} - \theta_3$. As before, let p_{ij} be as defined in (9), i.e., as the probability that the match (i, j) forms (and so, in this case, agrees) in the first period (and in any subsequent period that is reached). We also have that

$$p_{12} + p_{13} + p_{23} = 1. \tag{14}$$

Finally, from the definition of x_i, and substituting equation (12) for x_i, we have

$$v_{ij} - \theta_j = \delta \left[p_{ij} \frac{1}{2} (v_{ij} - \theta_j + \theta_i) + p_{ik} \frac{1}{2} (v_{ik} - \theta_k + \theta_i) \right],$$

which using (13) can be written as

$$\begin{aligned}
v_{12} - \theta_2 &= \delta \left(p_{12} + p_{13} \right) \frac{v_{12} + v_{13} - v_{23}}{2}, \\
v_{23} - \theta_3 &= \delta \left(p_{12} + p_{23} \right) \frac{v_{12} + v_{23} - v_{13}}{2}, \\
v_{13} - \theta_1 &= \delta \left(p_{13} + p_{23} \right) \frac{v_{13} + v_{23} - v_{12}}{2}.
\end{aligned}$$

This is a system of six equations with six unknowns, and its solutions would satisfy $x_1 = (p_{12} + p_{13}) \frac{v_{12}+v_{13}-v_{23}}{2}$. We know that player 1, conditional on meeting any of her potential partners, player 1 expects to obtain $\frac{v_{12}+v_{13}-v_{23}}{2}$. Let's use the notation A, B, C as above. We can indeed solve that system and obtain (11). For δ close to 1, all p's are positive and smaller than 1 only in Region 3, where $v_{12} < v_{13} + v_{23}$. Thus, for δ close enough to 1, this equilibrium could exist only in Region 3, and p_{ij} and x_i would coincide with the values in the equilibrium we have obtained in the existence part of the proof.

1.2) Now, suppose that (still $\delta x_i + \delta x_j < v_{ij}$ for all i, j) player i strictly prefers meeting player j rather than meeting player k when she is presented with a first invitation to meet in step 2. Taking Result 1 into account, two cases are possible without resulting in a cycle: either, a pair of players (say i and j) exists that prefer meeting with each other, so that

$$\begin{aligned} v_{ij} - \delta x_j &> v_{ik} - \delta x_k, \\ v_{ij} - \delta x_i &> v_{jk} - \delta x_k; \end{aligned} \tag{15}$$

or else, $v_{ij} - \delta x_i < v_{jk} - \delta x_k$ and $v_{ik} - \delta x_i = v_{jk} - \delta x_j$. Adding these latter two (in)equalities we contradict $v_{ij} - \delta x_j > v_{ik} - \delta x_k$. Thus, suppose (15) holds. That implies that $\lambda_i^k = \lambda_j^k = 0$ and that in turn requires that $\mu_i^k = \mu_j^k = 0$. Indeed, knowing that player j will certainly accept her first invitation to meet, player i strictly prefers to make player j a first invitation to meet when selected by Nature to do so: that guarantees the highest possible probability of a meeting for her and that results in the highest payoff. Similarly for player j. Therefore, if (15) holds, then $p_{ij} = 1$. Thus,

$$\begin{aligned} x_k &= 0, \\ x_i &= \frac{1}{2}(v_{ij} - \delta x_j + \delta x_i), \\ x_j &= \frac{1}{2}(v_{ij} - \delta x_i + \delta x_j), \end{aligned}$$

whose solution is $x_i = x_j = \frac{1}{2}v_{ij}$. This contradicts (15) for δ close to 1 unless $\frac{1}{2}v_{ij} \geq \max\{v_{ik}, v_{jk}\}$. (Region 1 with $i, j = 1, 2$.) Thus, we have found again the same equilibrium as in the previous proposition for Region 1, although this time it is only possible in the boundary case that $\frac{v_{12}}{2} = v_{13} = v_{23}$, otherwise we violate that $\delta x_i + \delta x_3 < v_{i3}$ for $i = 1, 2$ and for δ close to 1.

2) Now, consider an equilibrium where there exist a pair (i, j) such that $\delta x_i + \delta x_j \geq v_{ij}$.

2.1) Suppose that $\lambda_i^j > 0$, so that by Result 3 $x_i = 0$. Thus, $\delta x_j \geq v_{ij}$. This requires that $\delta x_k \geq v_{ik}$. Indeed, if $\delta x_k < v_{ik}$, then the expected payoff for i at step (4) if i and k meet is positive, $\frac{1}{2}(v_{ik} - \delta x_k)$. Therefore, a deviation to $\lambda_i^j = 0$ would be profitable for player i. But if $\delta x_k \geq v_{ik}$ and $\delta x_j \geq v_{ij}$ then $x_k > v_{ik}$ and $x_j > v_{ij}$, so that in equilibrium $\lambda_j^i = \lambda_k^i = 0$. Given these values of λ_j^i and λ_k^i, and since $x_k > v_{ik}$ and $x_j > v_{ij}$, equilibrium requires that $\mu_j^i = \mu_k^i = 0$. Thus, $p_{jk} = 1$. Therefore, $x_k = x_j = \frac{1}{2}v_{jk}$. Since $\delta x_k \geq v_{ik}$ and $\delta x_j \geq v_{ij}$, this can only happen in the interior of Region 1 ($v_{12} > 2v_{13}$), and coincides with our equilibrium obtained in the existence part of this proof.

2.2) Suppose that $\lambda_i^j = \lambda_j^i = 0$. Thus, $p_{ij} = 0$ and $\lambda_i^k = 1 - \lambda_i^j = 1 > 0$. Suppose that $x_i = 0$. If $\delta x_i + \delta x_k \geq v_{ik}$ then we can apply the same argument as in 2.1) above with (i, j) replaced by (i, k), and conclude that such equilibrium can only occur in the interior of Region 1 ($v_{12} > 2v_{13}$) and coincides with the one obtained in the existence part of this proof. On the other hand, $x_i = 0$ and $\delta x_i + \delta x_k < v_{ik}$ may only hold simultaneously if $p_{ik} = 0$. Otherwise, step (4) with a meeting of i and k occurs with positive probability and at that node the payoff of i is $\frac{1}{2}(v_{ik} - \delta x_i - \delta x_k) > 0$. In that case, $p_{jk} = 1$, and again the only such equilibrium is the one we obtained for Region 1. Similarly for player j.

Last, suppose that $x_i, x_j > 0$. Since $\lambda_i^j = \lambda_j^i = 0$, this requires that

$$v_{ik} - \delta x_i = v_{jk} - \delta x_j, \tag{16}$$

since otherwise either p_{ik} or p_{jk} would be equal to 1, so that either $x_i = 0$ or $x_j = 0$. Now, suppose $\delta x_i + \delta x_k \geq v_{ik}$. This implies that player i's expected payoff at step (4), if she meets k, is δx_i, and thus her expected payoff at step (1) is $p_{ik}\delta x_i$. Thus, $x_i \leq p_{ik}\delta x_i$, which is a contradiction to $x_i > 0$. The same holds for player j. Thus, $\delta x_i + \delta x_k < v_{ik}$ and $\delta x_j + \delta x_k < v_{jk}$, and hence from Result 2, $\rho_i^k = \rho_k^i = \rho_j^k = \rho_k^j = 1$. Thus, such equilibrium should satisfy

$$\begin{aligned} x_i &= p_{ik}\left(\frac{v_{ik} - \delta x_k + \delta x_i}{2}\right), \qquad (17) \\ x_j &= (1 - p_{ik})\left(\frac{v_{jk} - \delta x_k + \delta x_j}{2}\right), \\ x_k &= p_{ik}\left(\frac{v_{ik} - \delta x_i + \delta x_k}{2}\right) + (1 - p_{ik})\left(\frac{v_{jk} - \delta x_j + \delta x_k}{2}\right). \end{aligned}$$

Together with (16), this is a system of four equations that the four unknowns should satisfy in an equilibrium of this type. We study the solutions to this system as δ approaches 1. Substituting (16) in (17), and after some manipulation:

$$x_i + x_j = x_k(1 - \delta) + p_{ik}\delta x_i + (1 - p_{ik})\delta x_j.$$

Thus, as $\delta \to 1$ either p_{ik} and x_i both approach 0, or $1 - p_{ik}$ and x_j both approach 0. W.l.o.g, let the latter be the case. Note that substituting (16) in the third equation of (17) we have $(2 - \delta)x_k = v_{jk} - \delta x_j$, so that as $\delta \to 1$ and $x_j \to 0$, then $x_k \to v_{jk}$. And from the first equation in (17) then $x_i \to v_{ik} - v_{jk}$. Since we are assuming that $\delta x_i \geq v_{ij} - \delta x_j$, this requires for δ large enough that $v_{ik} - v_{jk} \geq v_{ij}$. Thus, this equilibrium can exist only in Region 2, for $k = 1$, $j = 2$ and $i = 3$ ($v_{12} \leq 2v_{13}$, and $v_{12} \leq v_{13} + v_{23}$). In other words, for δ large, this equilibrium would approach the one we already obtained in the existence part of this proof. QED

7.3 Proof of Proposition 3

Let i be chosen by Nature, and let (z_1, z_2, z_3) be the payoffs in a stationary subgame perfect equilibrium of $\widehat{\Gamma}(\delta, \alpha; \mathbf{x})$. A proposal by player i to divide V according to an offer (s_1, s_2, s_3) will be accepted only if

$$s_j \geq \delta\left[(1 - \alpha)z_j + \alpha x_j\right],$$

and rejected otherwise. In case of equality, both rejection and acceptance may be part of equilibrium. Thus, if players j's strategy, $j \neq i$, is to accept in case of equality, then the best response of player i is either to offer $s_j = \delta [(1-\alpha)z_j + \alpha x_j]$ to player $j \neq i$, or to make any offer that is otherwise rejected. (If some player j's strategy, $j \neq i$, implies rejection in case of indifference, then player i has no best response with an acceptable offer.) Thus, player i's offer in any equilibrium where her offer is accepted must satisfy

$$s_i = V - \delta [(1-\alpha)(z_j + z_k) + \alpha (x_j + x_k)]. \tag{18}$$

Therefore, in any equilibrium in which all offers are accepted, the payoffs satisfy

$$z_i = \frac{1}{3}s_i + \frac{2}{3}\delta [(1-\alpha)z_i + \alpha x_i].$$

Substituting for s_i, we can write the above equation as

$$z_i = \delta [(1-\alpha)z_i + \alpha x_i] + \frac{1}{3}\left[V - \delta \sum\nolimits_{l=1}^{3} ((1-\alpha)z_l + \alpha x_l)\right].$$

The last term is common for all players. Thus, we can simplify these three equations and obtain

$$(z_i - z_j)(1 - \delta(1-\alpha)) = \delta\alpha (x_i - x_j),$$

which together with $\sum_{l=1}^{3} z_l = V$ implies that:

$$z_i = \frac{1}{3}[V + A(\delta, \alpha)(2x_i - x_j - x_k)]$$

where $A(\delta, \alpha) \equiv \frac{\delta\alpha}{1-\delta(1-\alpha)}$. Note that, for any value of α, $A(\delta, \alpha) \to 1$ as $\delta \to 1$.

This is an equilibrium if $s_i > \delta z_i$, where s_i is defined in (18). This inequality may be written as

$$V - \delta \left[\sum\nolimits_{l=1}^{3} (1-\alpha)z_l + \alpha (z_i + x_j + x_k)\right] > 0,$$

which is satisfied since $\sum_{l=1}^{3} z_l = V$ and $z_l \geq x_l$ for $l = 1, 2, 3$. Thus, we only need to rule out an equilibrium where i has no profitable offer that will be accepted by j and k. That is, $s_i < \delta z_i$, where s_i is defined in (18). In that case,

$$z_i = \frac{1}{3}\delta z_i + \frac{2}{3}\delta [(1-\alpha)z_i + \alpha x_i],$$

which implies that $z_i < x_i$. This requires that $s_i < \delta x_i$, that is, $s_i - \delta x_i < 0$, or

$$V - \delta [(1-\alpha)(z_j + z_k) + \alpha (x_j + x_k + x_i)] < 0.$$

Since $x_j + x_k + x_i \leq V$ and $z_j + z_k \leq V$, this is a contradiction that concludes the proof. QED

7.4 Proof of Lemma 1

Without loss of generality, assume $v_i = 0$, for all $i = 1, 2, 3$. Assume the core is not empty, that is, condition (2) holds, and that x does not belong to the core. We will show that x does not belong to the BS of the grand coalition. We do not need to consider allocations where $x_i < 0$ for some i, or where $x_1 + x_2 + x_3 < V$, since they cannot be in the BS. Thus, assume that $x_i + x_j < v_{ij}$ for some i, j, so that $x_k > V - v_{ij}$, for $k \neq i, j$. Consider an objection y of i against k where $y_i + y_j = v_{ij}$, with $y_i > x_i$ and $y_j > x_j$. A counter-objection z of k against i should satisfy $z_j \geq y_j$, and $z_j + z_k = v_{jk}$, so that $z_k \leq v_{jk} - y_j = v_{jk} - (v_{ij} - y_i)$. Also, $z_k \geq x_k > V - v_{ij}$. Therefore, if

$$v_{jk} - (v_{ij} - y_i) < V - v_{ij},$$

or

$$y_i < V - v_{jk},$$

then there exists no counter-objection to objection y, and so x does not belong to the BS. If $x_i < V - v_{jk}$ we can always construct such y, and then a necessary condition for x to belong to the BS is that $x_i \geq V - v_{jk}$. Switching the subscripts i and j, we could consider an objection y' of j against k, and repeat the argument to show that a necessary condition for x to belong to the BS is that $x_j \geq V - v_{ik}$. Thus, a necessary condition is that

$$x_i + x_j \geq 2V - v_{jk} - v_{ik} \geq v_{ij},$$

where the last inequality follows from condition (2). This contradicts $x_i + x_j < v_{ij}$ and proves that the BS coincides with the core when the latter is not empty. Now assume that condition (2) is not satisfied. In particular, this implies that we are in Region 3. We have already shown that the $R-$solution belongs to the BS. Hence, we only need to show that any other allocation does not belong to the BS. Note that (2) implies that for any feasible allocation (including the efficient ones), if $x_i = U_i + \epsilon$ (in Region 3), then $x_j + x_k \leq v_{jk} - \epsilon$, for any $\epsilon > 0$. So, consider an efficient allocation such that this is the case for some ϵ, and an objection y of j against i, with $y_j = x_j + \frac{\epsilon}{2}$ and $y_k = v_{jk} - y_j = v_{jk} - x_j - \frac{\epsilon}{2}$. A counter-objection z of i against j should satisfy that $z_k \geq y_k$ but also $z_i \geq x_i$, so that $z_k \leq v_{ik} - x_i$. Thus, for i to indeed have a counter-objection against j it is required that

$$v_{ik} - x_i = v_{ik} - U_i - \epsilon \geq y_k = v_{jk} - x_j - \frac{\epsilon}{2},$$

that is, $x_j \geq v_{jk} - v_{ik} + U_i + \frac{\epsilon}{2} = U_j + \frac{\epsilon}{2}$, where the last equality follows from the definition of U_i. Thus, this is a necessary condition for x to be in the BS. Switching the subscripts j and k, we would also conclude that another necessary condition is that $x_k \geq U_k + \frac{\epsilon}{2}$. Thus, a necessary condition is that $x_i = V - x_j - x_k \leq V - U_j - U_k - \epsilon = U_i - \epsilon$. And this contradiction proves the result. QED