SGPE Summer School 2022 Mathematics Exam

Question 1 (20 points)

Let $f(x)$ and $h(x)$:

$$f(x) = \sqrt{2x + 5b^2} - ln(x)$$

$$h(x) = 64x^3 \times \left[\frac{y}{x}\sqrt{12b^2} + x^3 \right]$$

- Compute $h'(x)$, $f'(x)$, $h''(x)$ and $h''(x)$ [10 pt]

  Solution:

$$f(x) = (2x + 5b^2)^{\frac{1}{2}} - ln(x)$$

$$f'(x) = \frac{1}{2}(2x + 5b^2)^{-\frac{1}{2}}2 - \frac{1}{x}$$

$$f''(x) = -\frac{1}{2}(2x + 5b^2)^{-\frac{3}{2}}2 + \frac{1}{x^2}$$

$$h(x) = \frac{64x^3y\sqrt{12b^2}}{x} + 64x^6$$

$$h(x) = 128x^2yb\sqrt{3} + 64x^6$$

$$h'(x) = 256xyb\sqrt{3} + 384x^5$$

$$h''(x) = 256yb\sqrt{3} + 1920x^4$$

- Find the critical values for f(x) and for h(x) if $y = -8$ and $b = \sqrt{3}$ [10 pt]

  Solution:

$$f'(x) = \frac{1}{2}(2x + 5b^2)^{-\frac{1}{2}}2 - \frac{1}{x} = 0$$

$$(2x + 5b^2)^{-\frac{1}{2}} = \frac{1}{x}$$

$$\frac{1}{\sqrt{(2x + 5b^2)}} = \frac{1}{x}$$

$$\sqrt{(2x + 5b^2)} = x$$

$$2x + 5b^2 = x^2$$

After plugging in $b = \sqrt{3}$

$$x^2 - 2x - 5b^2 = 0; x^2 - 2x - 5(\sqrt{3})^2 = 0; x^2 - 2x - 15 = 0$$

$$x_{1,2} = \frac{2 \pm \sqrt{4 + 60}}{2} = \frac{2 \pm 8}{2} = 5 \ or \ -3$$

$$h'(x) = 256xyb\sqrt{3} + 384x^5 = 0$$

$$384x^5 = -256xyb\sqrt{3}$$

$$x^4 = \frac{-yb\sqrt{3}}{1.5}$$

After plugging in $y = -8$ and $b = \sqrt{3}$

$$x = [\pm 2]$$

$h'(x) = 0$ also when $x = 0$

$$h'(2) = 0$$

Question 2 (20 points)

Calculate the following limits

$$i)\ \lim_{x\to\frac{1}{4}} \sqrt{4x} - 5\ [2\ pt]$$

$$ii)\ \lim_{x\to+\infty} \frac{x^2-1}{x}\ [3\ pt]$$

$$iii)\ \lim_{x\to 1} \frac{x^2-1}{x^3-1}\ [3\ pt]$$

$$iv)\ \lim_{x\to 3^{\pm}} \frac{x^2-8x+12}{x^2-5x+6}\ [5\ pt]$$

$$v)\ \lim_{x\to -1^{+}} \frac{\sqrt{5x+9}-2}{(x+1)^2}\ [7\ pt]$$

Solution:

i)

$$\lim_{x\to\frac{1}{4}} \sqrt{4x} - 5$$
$$= \sqrt{4\frac{1}{4}} - 5$$
$$= -4$$

ii)

$$\lim_{x\to+\infty} \frac{x^2-1}{x}$$
$$x + \frac{1}{x}$$
$$= +\infty + 0$$
$$= +\infty$$

iii) Re-arrange the denominator factoring like we did in Quiz 1

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1}$$
$$= \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}$$
$$= \frac{(x+1)}{(x^2+x+1)} = \frac{1+1}{1^2+1+1} = \frac{2}{3}$$

iv) Re-arrange the denominator factoring like we did in Quiz 1

$$\lim_{x\to 3^{\pm}} \frac{x^2-8x+12}{x^2-5x+6}$$
$$= \frac{(x-2)(x-6)}{(x-3)(x-2)}$$
$$= \frac{(x-6)}{(x-3)}$$
$$\lim_{x\to 3^{+}} \frac{(x-6)}{(x-3)}$$
$$= \frac{(3.1-6)}{(3.1-3)} = -\infty$$
$$\lim_{x\to 3^{-}} \frac{(x-6)}{(x-3)}$$
$$= \frac{(2.9-6)}{(2.9-3)} = +\infty$$

iv) Undetermined $\frac{0}{0}$. Multiply and divide by $(\sqrt{5x+9}+2)$

$$\lim_{x\to -1^{+}} \frac{\sqrt{5x+9}-2}{(x+1)^2} \times \frac{(\sqrt{5x+9}+2)}{(\sqrt{5x+9}+2)}$$
$$= \frac{5x+9-4}{(x+1)^2(\sqrt{5x+9}+2)}$$
$$= \frac{5(x+1)}{(x+1)^2(\sqrt{5x+9}+2)}$$
$$= \frac{5}{(x+1)(\sqrt{5x+9}+2)}$$
$$= \lim_{x\to -1^{+}} \frac{5}{(x+1)} \lim_{x\to -1^{+}} \frac{1}{\sqrt{5x+9}+2}$$
$$= +\infty \frac{1}{4}$$
$$= +\infty$$

Question 3 (20 points)

Calculate the following integrals:

$$i) \int (7x+5)^{\frac{3}{2}}\ dx\ [2\ pt]$$

$$ii) \int \frac{e^x}{e^x+2}\ dx\ [3\ pt]$$

$$iii) \int_0^2 (\frac{2}{3}x^3 - 3x^3 + x^2)\ dx\ [2\ pt]$$

$$iv) \int x^2 ln(x)\ dx [5\ pt]$$

$$v) \int \frac{x^3}{\sqrt{9-x^2}}\ dx\ [8\ pt]$$

Solution: *i)* Use the method of u-substitution. So if $u = 7x+5$ then $\frac{du}{dx} = 7$; and $dx = \frac{1}{7}du$. Thus, re-write the above integral as $\int \frac{1}{7}u^{\frac{3}{2}}\ du$

$$\begin{aligned} \int \frac{1}{7}u^{\frac{3}{2}}\ du &= \frac{1}{7}\int u^{\frac{3}{2}}\ du \\ &= \frac{1}{7}\frac{1}{\frac{3}{2}+1}u^{\frac{3}{2}+1} + c \\ &= \frac{1}{7}\frac{1}{\frac{5}{2}}u^{\frac{5}{2}} + c \\ &= \frac{2}{5}\frac{1}{7}(7x+5)^{\frac{5}{2}} + c \end{aligned}$$

ii) Now set $u = e^x$, take ln on both sides and $ln(u) = x$, $\frac{dx}{du} = \frac{1}{u}$, so $dx = \frac{1}{u}du$. So

$$\begin{aligned} \int \frac{e^x}{e^x+2}\ dx &= \int \frac{u \times \frac{1}{u}}{u+2}\ du \\ &= \int \frac{1}{u+2}\ du \\ &= ln(u+2) + c \\ &= ln(e^x+2) + c \end{aligned}$$

iii) Integral of a sum is the sum of the integrals

$$\begin{aligned} \int_0^2 (\frac{2}{3}x^3 - 3x^3 + x^2)\ dx &= \int_0^2 \frac{2}{3}x^3 - \int_0^2 3x^3 + \int_0^2 x^2 \\ &= [\frac{2}{12}x^4]_0^2 - [\frac{3}{4}x^4]_0^2 + [\frac{1}{3}x^3]_0^2 \\ &= \frac{2}{12}16 - \frac{3}{4}16 + \frac{1}{3}8 \\ &= -\frac{19}{3} \end{aligned}$$

iv) Use integration by parts. Remember $\int udv = uv - \int vdu$. Let $u = ln(x)$, $dv = x^2$, $v = \frac{1}{3}x^3$ and $du = \frac{1}{x}$ then solve

$$\begin{aligned}\int x^2 ln(x)\ dx &= ln(x)\frac{1}{3}x^3 - \int \frac{1}{3}x^3\frac{1}{x} \\ &= ln(x)\frac{1}{3}x^3 - \frac{1}{3}\int x^2 \\ &= ln(x)\frac{1}{3}x^3 - \frac{1}{3}\frac{1}{3}x^3 + c \\ &= (ln(x) - \frac{1}{3})\frac{1}{3}x^3 + c\end{aligned}$$

v) Use the method of u-substitution. So if $u = 9 - x^2$ then $\frac{du}{dx} = -2x$; $-\frac{du}{2} = xdx$. Thus, re-write the above integral as

$$\begin{aligned}\int \frac{x^2 x}{\sqrt{9-x^2}}dx &= \int \frac{u-9}{2\sqrt{u}}\ du \\ &= \int \frac{u}{2\sqrt{u}}du - \frac{9}{2}\int \frac{1}{\sqrt{u}}du \\ &= \frac{1}{2}\int \sqrt{u}du - \frac{9}{2}\int \frac{1}{\sqrt{u}}du \\ &= \frac{1}{2}\frac{2}{3}u\sqrt{u} - \frac{9}{2}2\sqrt{u} + c \\ &= \frac{1}{3}u\sqrt{u} - 9\sqrt{u} + c \\ &= \frac{1}{3}\sqrt{u}(u - 27) + c \\ &= \frac{1}{3}\sqrt{9-x^2}(9 - x^2 - 27) + c \\ &= \frac{1}{3}\sqrt{9-x^2}(-18 - x^2) + c\end{aligned}$$

Question 4 (20 points)

(i) Given the function $f(x) = x^2 - 4x - 12$, find 1) the critical values, 2) at which value of $f(.)$ this is optimised, 3) whether this is a relative maximum or minimum using the second-order condition. [5 pt]

(ii) Given the function $f(x) = 9(x-3)^3 + 2$, find 1) the critical values, 2) at which value of $f(.)$ this is optimised, 3) whether this is a relative maximum or minimum using the second-order condition. [5 pt]

(iii) Given the production function $F(K,L) = 3K^{\frac{1}{3}}L^{\frac{2}{3}}$ where input K is capital and input L is labour, find the marginal product of each input. *Hint: you need to take the partial derivative with respect to each variable*. Given the cost function $K + 2L = 600$ find the amount of labour and capital that maximise production. *Hint: you can transform this problem into one of optimisation in only one variable.* [10 pt]

Solution: *i)* First-order condition $f'(x) = 0; 2x - 4 = 0; x = 2$ so the function is optimised at $f(2) = 2^2 - 4 \times 2 - 12 = -16$. Second-order condition $f''(x) = 2 > 0$ so the point is a relative minimum.
ii) First-order condition $f'(x) = 0; 27(x-3)^2 = 0$. This is an equation of order 2 that admits only one solution, i.e. $x = 3$ so the function is optimised at $f(3) = 2$. Second-order condition $f''(x) = 54(x-3)$ is 0 when $x = 3$ so this is an inflection point.
iii)

$$MP_K = \frac{\partial F(K,L)}{\partial K} = K^{-\frac{2}{3}}L^{\frac{2}{3}} = \left(\frac{L}{K}\right)^{\frac{2}{3}}$$

$$MP_L = \frac{\partial F(K,L)}{\partial L} = K^{\frac{1}{3}}L^{-\frac{1}{3}} = \left(\frac{K}{L}\right)^{\frac{1}{3}}$$

Set the cost cost function in terms of one of the two inputs $K = 600 - 2L$ and plug this in the production function. This will give as a function in one variable. Then we can maximise with respect to that variable.

$$3K^{\frac{1}{3}}L^{\frac{2}{3}} = 3(600 - 2L)^{\frac{1}{3}}L^{\frac{2}{3}}$$

Take the first derivative and set equal to zero

$$\begin{aligned}
\frac{1}{3}3(600-2L)^{-\frac{2}{3}}(-2)L^{\frac{2}{3}} + \frac{2}{3}3(600-2L)^{\frac{1}{3}}L^{-\frac{1}{3}} &= 0 \\
2(600-2L)^{-\frac{2}{3}}L^{\frac{2}{3}} &= 2(600-2L)^{\frac{1}{3}}L^{-\frac{1}{3}} \\
\frac{L^{\frac{2}{3}}}{L^{-\frac{1}{3}}} &= \frac{2(600-2L)^{\frac{1}{3}}}{2(600-2L)^{-\frac{2}{3}}} \\
L &= 600 - 2L \\
L^* &= 200 \\
K^* &= 600 - 2L^* \\
K^* &= 200
\end{aligned}$$

Question 5 (20 points)

Consider the following matrices and perform the required operations where possible.

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix}; B = \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 0 & 3 \end{pmatrix}; C = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}; D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

(i) calculate $A \times B$, $B \times A$, $A \times C$, $C \times B$ and $C \times A$ [2 pt]

(ii) find $A \times B$, $B \times A$, $A \times C$, $B \times C$ and $C \times A$ determinants [5 pt]

(iii) find $A \times B$, $B \times A$, $A \times C$, $B \times C$ and $C \times A$ inverses [7 pt]

(iv) find the inverse of D [6 pt]

Solution: *i)*

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix} \times B = \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 0 & 3 \end{pmatrix}$$

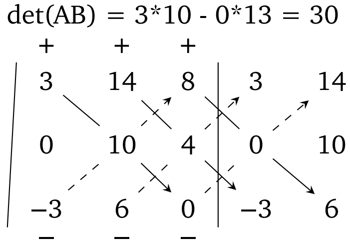
$$B = \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 0 & 3 \end{pmatrix} \times A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 14 & 8 \\ 0 & 10 & 4 \\ -3 & 6 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \times A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 4 \\ -1 & 7 & 2 \end{pmatrix}$$

Students should notice that $A \times C$ and $C \times B$ are not possible

ii) Students should notice that determinants can only be calculated for square matrices.

det(AB) = 3*10 - 0*13 = 30



det(BA) = 0 - 168 + 0 - (0 + 72 - 240) = 0

iii) Only squared matrix have a determinant, which is a necessary, yet not sufficient condition for them to be inverted.

$$AB^{-1} = \frac{1}{30}\begin{pmatrix} 3 & -13 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} \frac{3}{30} & -\frac{13}{30} \\ 0 & \frac{10}{30} \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & -\frac{13}{30} \\ 0 & \frac{1}{3} \end{pmatrix}$$

det(BA)= 0 so this matrix is not invertible.

iv)

det(D)=1

$$cofD = \begin{pmatrix} 1 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}; adjD = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}; D^{-1} = \frac{1}{det(D)}\begin{pmatrix} 1 & 0 & 0 \\ -4 & 2 & -1 \\ 3 & -1 & 1 \end{pmatrix}$$