SGPE Summer School 2023 Mathematics Exam

Question 1 (20 points)

Let $f(x)$ and $h(x)$:

$$f(x) = \sqrt{(2x + 3\sqrt{b})} - 2ln(x)$$

$$h(x) = bx^2 \left[\frac{y}{x} \sqrt[3]{4b} + x^3 \right]$$

- Compute $f'(x)$, $h'(x)$, $f''(x)$ and $h''(x)$ [10 pt]

  Solution:

$$f(x) = (2x + 3\sqrt{b})^{\frac{1}{2}} - 2ln(x)$$

$$f'(x) = \frac{1}{\sqrt{(2x + 3\sqrt{b})}} - \frac{2}{x}$$

$$f''(x) = -\frac{1}{(2x + 3\sqrt{b})^{\frac{3}{2}}} + \frac{2}{x^2}$$

$$h(x) = by\sqrt[3]{4b}x + bx^5$$

$$h'(x) = y\sqrt[3]{4}b^{\frac{4}{3}} + 5bx^4$$

$$h''(x) = 20bx^3$$

- Find the critical values for f(x) and for h(x) if $y = -3$ and $b = 16$ [10 pt]

  Solution:

$$f'(x) = \frac{1}{\sqrt{(2x + 3\sqrt{b})}} - \frac{2}{x} = 0$$

$$\frac{1}{\sqrt{(2x + 3\sqrt{b})}} = \frac{2}{x}$$

After plugging in $b = 16$

$$x = 2\sqrt{(2x + 12)}$$

$$x^2 = 4(2x + 12)$$

$$x^2 - 8x - 48 = 0$$

$$x_{1,2} = \frac{8 \pm \sqrt{64 + 192}}{2} = \frac{8 \pm 16}{2} = 12\ or\ -4\ (although\ -4\ is\ not\ part\ of\ the\ function's\ domain)$$

$$h'(x) = by\sqrt[3]{4b} + 5bx^4 = 0$$

$$x^4 = -\frac{y\sqrt[3]{4b}}{5}$$

After plugging in $y = -3$ and $b = 16$

$$x^4 = \frac{12}{5}$$

$$x = \pm\sqrt[4]{\frac{12}{5}}$$

Question 2 (20 points)

Calculate the following limits

$$i)\ \lim_{x\to 1} e^{2x} + 1\ [2\ pt]$$
$$ii)\ \lim_{x\to -\infty} \frac{x^4 - 1}{x}\ [5\ pt]$$
$$iii)\ \lim_{x\to 2} \frac{3x^2 - 7x + 2}{x^2 + x - 6}\ [6\ pt]$$
$$iv)\ \lim_{x\to 3} \frac{\sqrt{3x} - 3}{\sqrt{2x-4} - \sqrt{2}}\ 7\ pt]$$

Solution:

i) Function is continuous at 2, so plug in

$$\lim_{x\to 1} e^{2x} + 1$$
$$= e^{2\cdot 1} + 1$$
$$= e^2 + 1$$

ii)

$$\lim_{x\to -\infty} \frac{x^4 - 1}{x}$$
$$\lim_{x\to -\infty} \left(x^3 - \frac{1}{x}\right)$$
$$= -\infty - 0$$
$$= -\infty$$

iii) Attempting direct substitution gives $\frac{0}{0}$, so we factor

$$\lim_{x\to 2} \frac{3x^2 - 7x + 2}{x^2 + x - 6}$$
$$= \lim_{x\to 2} \frac{(3x-1)(x-2)}{(x+3)(x-2)}$$
$$= \lim_{x\to 2} \frac{(3x-1)}{(x+3)}$$
$$= \frac{(6-1)}{(2+3)} = 1$$

iv) Undetermined $\frac{0}{0}$. Multiply and divide by $\sqrt{3x} + 3$

$$\lim_{x\to 3} \frac{\sqrt{3x} - 3}{\sqrt{2x-4} - \sqrt{2}} \times \frac{(\sqrt{3x} + 3)}{(\sqrt{3x} + 3)}$$
$$= \frac{3x - 9}{(\sqrt{2x-4} - \sqrt{2})(\sqrt{3x} + 3)}$$

Now multiply and divide by $\sqrt{2x-4}+\sqrt{2}$

$$\begin{aligned}
&\lim_{x\to 3}\frac{3x-9}{(\sqrt{2x-4}-\sqrt{2})(\sqrt{3x}+3)}\times\frac{(\sqrt{2x-4}+\sqrt{2})}{(\sqrt{2x-4}+\sqrt{2})}\\
&=\lim_{x\to 3}\frac{(3x-9)(\sqrt{2x-4}+\sqrt{2})}{(\sqrt{2x-4}+\sqrt{2})(\sqrt{2x-4}-\sqrt{2})(\sqrt{3x}+3)}\\
&=\lim_{x\to 3}\frac{(3x-9)(\sqrt{2x-4}+\sqrt{2})}{(2x-6)(\sqrt{3x}+3)}\\
&=\lim_{x\to 3}\frac{3(x-3)(\sqrt{2x-4}+\sqrt{2})}{2(x-3)(\sqrt{3x}+3)}\\
&=\lim_{x\to 3}\frac{3(\sqrt{2x-4}+\sqrt{2})}{2(\sqrt{3x}+3)}\\
&\qquad=\frac{3(\sqrt{2}+\sqrt{2})}{2(3+3)}=\frac{\sqrt{2}}{2}
\end{aligned}$$

Question 3 (20 points)

Calculate the following integrals:

$$i) \int (5x+3)^{\frac{5}{4}}\ dx\ [2\ pt]$$

$$ii) \int \frac{3e^x}{5e^x+1}\ dx\ [3\ pt]$$

$$iii) \int_0^1 (\frac{5}{4}x^4 - 2x^2 + x)\ dx\ [2\ pt]$$

$$iv) \int (x^2+1)ln(x)\ dx[5\ pt]$$

$$v) \int \frac{3x+2}{x^2+5x+6}\ dx\ [8\ pt]$$

Solution: *i)* Use the method of u-substitution. So if $u = 5x+3$ then $\frac{du}{dx} = 5$; and $dx = \frac{1}{5}du$. Thus, re-write the above integral as $\int \frac{1}{5}u^{\frac{5}{4}}\ du$

$$\begin{aligned}\int \frac{1}{5}u^{\frac{5}{4}}\ du &= \frac{1}{5}\int u^{\frac{5}{4}}\ du \\ &= \frac{1}{5}\frac{1}{\frac{5}{4}+1}u^{\frac{5}{4}+1} + c \\ &= \frac{1}{5}\frac{1}{\frac{9}{4}}u^{\frac{9}{4}} + c \\ &= \frac{1}{5}\frac{4}{9}(5x+3)^{\frac{9}{4}} + c = \frac{4}{45}(5x+3)^{\frac{9}{4}} + c\end{aligned}$$

ii) Rewrite the integral as:

$$\int \frac{3e^x}{5e^x+1}\ dx = \frac{3}{5}\int \frac{5e^x}{5e^x+1}\ dx$$

Now set $u = 5e^x + 1$, then $\frac{du}{dx} = 5e^4$; and $dx = 5e^x du$.

$$\begin{aligned}\frac{3}{5}\int \frac{5e^x}{5e^x+1}\ dx &= \frac{3}{5}\int \frac{1}{u}\ dx \\ &= \frac{3}{5}ln(u) + c \\ &= \frac{3}{5}ln(5e^x+1) + c\end{aligned}$$

iii) Integral of a sum is the sum of the integrals

$$\int_0^1 \left(\frac{5}{4}x^4 - 2x^2 + x\right)\, dx = \int_0^1 \frac{5}{4}x^4\, dx - \int_0^1 2x^2\, dx + \int_0^1 x\, dx$$
$$= \left[\frac{5}{4}\frac{1}{5}x^5 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1$$
$$= \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 0 + 0 - 0 = \frac{1}{12}$$

iv) Use integration by parts. Remember $\int udv = uv - \int vdu$. Let $u = ln(x)$, $v = \frac{x^3}{3} + x$ so you can rewrite the integral as:

$$\int (x^2+1)ln(x)\, dx = \int ln(x)\, d\left[\frac{x^3}{3} + x\right]$$
$$= ln(x)\left(\frac{x^3}{3} + x\right) - \int \left(\frac{x^3}{3} + x\right)\, d\,(ln(x))$$
$$= ln(x)\left(\frac{x^3}{3} + x\right) - \int \left(\frac{x^3}{3} + x\right)\frac{1}{x}\, dx$$
$$= ln(x)\left(\frac{x^3}{3} + x\right) - \int \left(\frac{x^2}{3} + 1\right)\, dx$$
$$= ln(x)\left(\frac{x^3}{3} + x\right) - \frac{x^3}{9} - x + c$$

v) Use partial fractions. Before attempting integration, we look for numbers ***A*** and ***B*** such that:

$$\frac{3x+2}{x^2+5x+6} = \frac{3x+2}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

So we have:

$$\frac{A}{x+2} + \frac{B}{x+3} = \frac{A(x+3) + B(x+2)}{(x+2)(x+3)} = \frac{(A+B)x + 3A + 2B}{(x+2)(x+3)}$$

Compare this to the original integrand:

$$\frac{(A+B)x + 3A + 2B}{(x+2)(x+3)} = \frac{3x+2}{(x+2)(x+3)}$$

Therefore:

$$A + B = 3$$
$$3A + 2B = 2$$

This simple system of equations gives us $A = -4$ and $b = 7$, so the integral can be rewritten as:

$$\begin{aligned}\int \frac{3x+2}{x^2+5x+6}\,dx &= \int \left(-\frac{4}{x+2} + \frac{7}{x+3}\right)\,dx \\ &= -4ln(x+2) + 7ln(x+3) + c\end{aligned}$$

Question 4 (20 points)

(i) Given the production function $F(K, L) = K^\alpha L^{1-\alpha}$ where input K is capital and input L is labour, find the marginal product of each input. *Hint: you need to take the partial derivative with respect to K and L.* [10pt]

(ii) Using the same production function $F(K, L) = K^\alpha L^{1-\alpha}$ and given a production budget B, the firm will spend it on inputs such that $rK + wL = B$, where r and w are prices of inputs. Find the amount of labour and capital that maximise production, as functions of the parameters α, w, r and B. *Hint: you can turn this into a maximisation problem in only one variable. Also, you can treat parameters as if they were given numbers. Your unknowns are K and L.* [10 pt]

Solution:

i)

$$MP_K = \frac{\partial F(K, L)}{\partial K} = \alpha K^{\alpha-1} L^{1-\alpha} = \alpha \left(\frac{L}{K}\right)^{1-\alpha}$$

$$MP_L = \frac{\partial F(K, L)}{\partial L} = (1-\alpha) K^\alpha L^{-\alpha} = (1-\alpha) \left(\frac{K}{L}\right)^{\alpha}$$

ii) Express K as a function of L in the budget constraint:

$$K = \frac{B}{r} - \frac{w}{r} L$$

Plug this into the production function:

$$\tilde{F}(L) = \left(\frac{B}{r} - \frac{w}{r} L\right)^\alpha L^{1-\alpha}$$

Take the first order condition with respect to L:

$$\frac{d\tilde{F}}{dL} = \alpha \left(\frac{B}{r} - \frac{w}{r} L\right)^{\alpha-1} \left(-\frac{w}{r}\right) L^{1-\alpha} + \left(\frac{B}{r} - \frac{w}{r} L\right)^\alpha (1-\alpha) L^{-\alpha} = 0$$

$$\frac{\alpha w}{r} \left(\frac{B}{r} - \frac{w}{r} L\right)^{\alpha-1} L^{1-\alpha} = \left(\frac{B}{r} - \frac{w}{r} L\right)^\alpha (1-\alpha) L^{-\alpha}$$

$$\frac{\alpha w}{r} L = (1-\alpha) \left(\frac{B}{r} - \frac{w}{r} L\right)$$

$$L = (1-\alpha) \frac{B}{w}$$

Substitute this back into the budget constraint to find K:

$$K = \alpha \frac{B}{r}$$

Question 5 (20 points)

Consider the following matrices and perform the required operations where possible.

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 2 & -2 & 4 \end{pmatrix}; B = \begin{pmatrix} 3 & -4 \\ 1 & 2 \\ -2 & 4 \end{pmatrix}; C = \begin{pmatrix} -3 & 1 \\ 2 & 4 \end{pmatrix}; D = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

(i) calculate $A \times B$, $B \times A$, $A \times C$, $C \times B$ and $C \times A$ [2 pt]

(ii) find $A \times B$, $B \times A$, $A \times C$, $B \times C$ and $C \times A$ determinants [5 pt]

(iii) find $A \times B$, $B \times A$, $A \times C$, $B \times C$ and $C \times A$ inverses [7 pt]

(iv) find the inverse of D [6 pt]

Solution: *i)*

$$A \times B = \begin{pmatrix} 15 & -8 \\ -4 & 4 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 1 & 20 & -19 \\ 7 & 0 & 7 \\ 2 & -16 & 18 \end{pmatrix}$$

$$C \times A = \begin{pmatrix} -7 & -14 & 7 \\ 14 & 0 & 14 \end{pmatrix}$$

Students should notice that $A \times C$ and $C \times B$ are not possible

ii) Students should notice that determinants can only be calculated for square matrices.

$det(AB) = 15 \cdot 4 - (-4)(-8) = 28$

$$\begin{array}{ccc|cc} + & + & + & & \\ 1 & 20 & -19 & 1 & 20 \\ 7 & 0 & 7 & 7 & 0 \\ 2 & -16 & 18 & 2 & -16 \\ - & - & - & & \end{array}$$

$det(BA) = 1 \cdot 0 \cdot 18 + 20 \cdot 7 \cdot 2 + (-19) \cdot 7 \cdot (-16) - 2 \cdot 0 \cdot (-19) - (-16) \cdot 7 \cdot 1 - 18 \cdot 7 \cdot 20 = 0$

iii) Only square matrices have a determinant, which is a necessary, yet not sufficient condition for them to be inverted.

$$AB^{-1} = \frac{1}{28} \begin{pmatrix} 4 & 8 \\ 4 & 15 \end{pmatrix} = \begin{pmatrix} \frac{4}{28} & \frac{8}{28} \\ \frac{4}{28} & \frac{15}{28} \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{1}{7} & \frac{15}{28} \end{pmatrix}$$

det(BA)$=0$ so this matrix is not invertible.

iv)

det(D)=-5

$$cofD = \begin{pmatrix} -1 & -2 & 0 \\ -5 & 0 & -5 \\ 2 & -1 & 0 \end{pmatrix}; adjD = \begin{pmatrix} -1 & -5 & 2 \\ -2 & 0 & -1 \\ 0 & -5 & 0 \end{pmatrix}$$

$$D^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -5 & 2 \\ -2 & 0 & -1 \\ 0 & -5 & 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 5 & -2 \\ 2 & 0 & 1 \\ 0 & 5 & 0 \end{pmatrix}$$