UoE - SGPE Economics Summer School (2019) Mathematics Examination

July 2019

Question 1

Let f(g):

$$f(g) = ln(g^3) \times e^g$$

- Compute $f'(g)$ and $f''(g)$ [5 pt]

  Solution:

$$f'(g) = e^g(\frac{3}{g} + ln\ g^3)$$

$$f''(g) = \frac{e^g g^2 ln\ g^3 + 6e^g g - 3e^g}{g^2}$$

Where:

$$\frac{\mathrm{d}}{\mathrm{d}g}(\frac{3e^g}{g}) = 3(\frac{e^g g - e^g}{g^2})$$

$$\frac{\mathrm{d}}{\mathrm{d}g}(e^g ln\ g^3) = e^g ln\ g^3 + \frac{3e^g}{g}$$

Question 2

$$G(x) = \frac{32x^2 + 40ln(e^x)}{x^{-1}}$$

- Find the critical values for G(x) [5pt]

Solution:

$$G'(x) = 96x^2 + 80x = 0$$

$$X = [0, \frac{-5}{6}]$$

$$G'(0) = 0$$

$$G'(\frac{-5}{6}) = 0$$

Question 3

$$\textbf{(a)} \lim_{x\to 1} \frac{ln(x^x)}{\sqrt{x+8}}[4pt]$$

Solution:

Apply L'H$\hat{o}$pital's rule: $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$

$f'(x) = 1$ since, $ln(x^x) = xln(x)$ then $f'(x) = ln(x) + 1$

$g'(x) = \frac{-1}{2(x+8)^{\frac{3}{2}}}$

$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \frac{1}{\frac{-1}{2(x+8)^{\frac{3}{2}}}} = -54$

$$\textbf{(b)} \lim_{x\to \infty} \frac{x^2+18x+81}{x+9}[3pt]$$

Solution:

Simplify, $x^2 + 18x + 81 = (x + 9)^2$

Then $\frac{(x+9)^2}{x+9} = x + 9$
$\lim_{x\to\infty} = \infty$

$$\textbf{(c)} \lim_{x\to \infty} \frac{\sqrt{x}-8}{x-64}[3pt]$$

Solution:

Multiply and divide by $(\sqrt{x} + 8)$: $\lim_{x\to\infty} \frac{(\sqrt{x}-8)(\sqrt{x}+8)}{(x-64)(\sqrt{x}+8)}$

$\lim_{x\to\infty} \frac{(x-64)}{(x-64)(\sqrt{x}+8)}$

$\lim_{x\to\infty} \frac{1}{(\sqrt{x}+8)} = 0$

Question 4

(a) Consider the following matrix A:

$$\underset{(m\times n)}{A} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

Find: A^T [3pt], where A^T is the transpose of the matrix A. Also, find $(A^T)^T$ [2pt].

(**Hint**: You don't need to write every element in the matrix, a matrix with a number of explicit elements similar to that of matrix A, will suffice.)

Solution:

$$\underset{(n\times m)}{A^T} = \begin{bmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{1n} & a_{2n} & ... & a_{mn} \end{bmatrix}$$

$$\underset{(m\times n)}{(A^T)^T} = \underset{(m\times n)}{A}$$

(b) Consider the following matrices:

$$\underset{(2\times 3)}{Q} = \begin{bmatrix} 1 & 0 & 3 \\ 8 & 4 & -1 \end{bmatrix}$$

$$\underset{(3\times 2)}{L} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Find: QL [3pt], where QL is the product of both matrices. Also, find $(QL)^{-1}$, where $(QL)^{-1}$ is the inverse of QL [2pt].

Solution:

$$\underset{(2\times2)}{QL} = \begin{bmatrix} (1*3)+(0*3)+(3*1) & (1*4)+(0*2)+(3*1) \\ (8*3)+(4*3)+(-1*1) & (8*4)+(4*2)+(-1*1) \end{bmatrix}$$

$$\underset{(2\times2)}{QL} = \begin{bmatrix} 6 & 7 \\ 35 & 39 \end{bmatrix}$$

$$\underset{(2\times2)}{QL^{-1}} = \frac{1}{-11} \times \begin{bmatrix} 39 & -7 \\ -35 & 6 \end{bmatrix}$$

$$\underset{(2\times2)}{QL^{-1}} = \begin{bmatrix} \frac{-39}{11} & \frac{7}{11} \\ \frac{35}{11} & \frac{-6}{11} \end{bmatrix}$$

Question 5

(a) Consider the following production function for a manufacturing company:

$$Q = 5L^{0.5}K - L$$

Where, Q represents units produced, L represents labour and K represents capital.

If capital is fixed at 10 units, find the number of employees that maximizes production [5pt]. (**Hint:** Production is maximised when the marginal product of labour (MPL) is equal to zero. The MPL is found by calculating $\frac{\partial Q}{\partial L}$)

Solution:

$Q = 50L^{0.5} - L$

$\frac{\partial Q}{\partial L} = MPL = 25L^{-0.5} - 1 = 0$

$MPL = L^{-0.5} = \frac{1}{25}$

$(L^{-0.5})^{-2} = (\frac{1}{25})^{-2}$

$L^* = 625$

If capital is fixed at 10, production is maximised when the company hires 625 employees and produces 625 units. Any other number of employees, larger or smaller will correspond to a smaller overall production.

(b) The company will follow your advice and hire an additional 100 employees to reach the target specified for production maximization. The company is short on cash and requires a loan of $10,000£$. The company requests a loan of $10,000$ pounds today and the interest rate for the loan is r. Find the interest rate r if the total amount owed after (n=10) years is equal to $200,000$ pounds (Note: The sum was continuously compounded for n years.). [5pt]

Solution:

$10,000 * e^{10*r} = 200,000$

$e^{10*r} = 20$

take ln of both sides:

$ln(e^{10r}) = ln(20)$

$10r = 2.995732$

$r = 0.2995732$

Question 6

Solve the following linear system of equations [10pt]:

$$\begin{aligned} 3x - 3y + 2z &= -21 \\ 21x - y &= -7 \\ 7x + 4y - 3z &= 10 \\ 30x - 10y + 6z &= -70 \end{aligned}$$

Solution:

Note that the last equation is a linear combination of the first two equations so it is redundant.

Re-arranging the second equation first:

$y = 21x + 7$

Then sub into equations one and three:

$3x - 3(21x + 7) + 2z = -21$

$z = 30x$

And

$7x + 4(21x + 7) - 3z = 10$

$91x - 3z = -18$

Now, using these last reduced equation, we use the second equation to replace for z:

$91x - 3(30x) = -18$

$x = -18$

Similarly:

$z = 30(-18)$

$z = -540$

And

$y = 21(-18) + 7$

$y = -371$

So that:

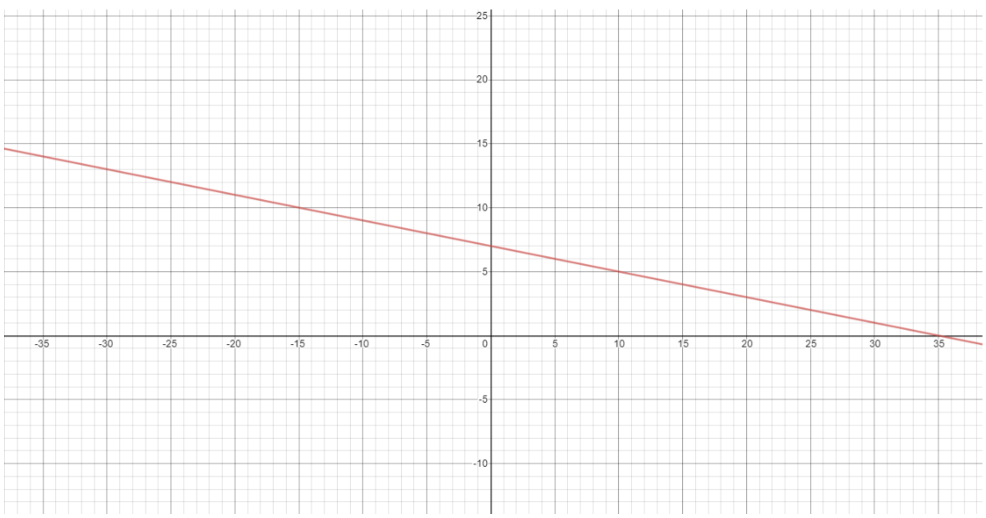
$x = -18; \;\; y = -371; \;\; z = -540$

Question 7

(a) Graph the following equation:

$y = -0.2x + 7$ [3pt]:

Solution:[1]



(b) Determine which of the following relations are functions. Specify the domain and range. Also mention if the function is continuous.

(1) $y^2 = x + 2$ [2pt]:

Solution:

for $x = 2$, y has two values, $y = 2$ and $y = -2$ which is not a unique outcome, consequently is not a function.

(2) $y = ln(x)$ [2pt]:

Solution:

The function has a domain $(0, \infty)$. The range is the real numbers ($\mathbb{R}$) and the function is continious.

(c) Calculate the following integral [3pt]:

[1] Original problem from the Khan Academy practice guide.

$$\int x(x+1)^{\frac{3}{2}}\,dx$$

Solution:

We can use integration by parts:

$u = x$ and $dv = (x+1)^{\frac{3}{2}}\,dx$

And calculate:

$du = dx$ and $v = \frac{2}{5}(x+1)^{\frac{5}{2}}$

Then using the integration by parts formula:

$\int x(x+1)^{\frac{3}{2}}\,dx = \frac{2}{5}x(x+1)^{\frac{5}{2}} - \frac{2}{5}\int (x+1)^{\frac{5}{2}}\,dx$

Then, calculating the integral on the right hand side:

$\int x(x+1)^{\frac{3}{2}}\,dx = \frac{2}{5}x(x+1)^{\frac{5}{2}} - \frac{4}{35}(x+1)^{\frac{7}{2}} + c$

Question 8

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sqrt{x}}$.

Find the general expression for the k^{th} derivative $f^{(k)}(x)$ [7pt]. Also, find the 5^{th} derivative $f^{(5)}(x)$ [3pt]:

[**Hint:** Write the general expression first and use it to compute the 5^{th} derivative. The notation $\prod$ is used to represent the product of a sequence, for example: $\prod_{i=0}^{n}(x - i) = (x - 0) \times (x - 1) \times ... \times (x - n)$]

Solution:

$f(x) = x^{\frac{-1}{2}}$

$f'(x) = (\frac{-1}{2})x^{(\frac{-1}{2}-1)}$

$f''(x) = (\frac{-1}{2})(\frac{-1}{2} - 1)x^{(\frac{-1}{2}-2)}$

$f'''(x) = (\frac{-1}{2})(\frac{-1}{2} - 1)(\frac{-1}{2} - 2)x^{(\frac{-1}{2}-3)}$

Then the k^{th} derivative would be:

$f^{(k)}(x) = \left[\prod_{i=0}^{k-1}(\frac{-1}{2} - i) \right] x^{(\frac{-1}{2}-k)}$

The 5^{th} derivative would be:

$f^{(5)}(x) = (-0.5-0)\times(-0.5-1)\times(-0.5-2)\times(-0.5-3)\times(-0.5-4)x^{(-0.5-5)}$

$f^{(5)}(x) = 29.53x^{(\frac{-11}{2})}$