UoE - Summer School 2017 - Mathematics

July 2017

Question 1

Let G(x):

$$G(x) = \frac{3x^2 - 27ln(e^x)}{x^{-2}}$$

- Compute $G'(x)$ and $G''(x)$ [6 pt]

  Solution:

$$G(x) = \ 3x^3(x-9)$$

$$G'(x) = \ 12x^3 - 81x^2$$

$$G'(x) = \ 3x^2(4x-27)$$

$$G''(x) = \ 36x^2 - 162x$$

$$G''(x) = \ 3x(12x-54)$$

- Find the critical values of G(x). [4pt]

  Solution:

$$G'(x) = \ 3x^2(4x-27) = 0$$

$$X = [0, 6.75]$$

$$G'(0) = 0$$

$$G'(27/4) = 0$$

Question 2

Evaluate the following limits:

$$\textbf{(a)} \lim_{x\to 1} \frac{x^2 - 8x + 15}{x - 3} [2pt]$$

Solution:

$\lim_{x\to 1} \frac{(x-3)(x-5)}{(x-3)} = -4$

$$\textbf{(b)} \lim_{x\to\infty} \frac{sin(x)}{x})[3pt]$$

Solution:

Note: $-1 <= sin(x) <= +1$

Then, $\lim_{x\to\infty} \frac{sin(x)}{x}$

Apply squeeze theorem: $\frac{-1}{x} <= \frac{sin(x)}{x} <= \frac{1}{x}$

By the squeeze theorem: $\lim_{x\to\infty} \frac{-1}{x} <= \frac{sin(x)}{x} <= \frac{1}{x}$

Since: $\lim_{x\to\infty} \frac{sin(-1)}{x} = 0$

And: $\lim_{x\to\infty} \frac{sin(1)}{x} = 0$

Then: $\lim_{x\to\infty} \frac{sin(x)}{x} = 0$

$$\textbf{(c)} \lim_{x\to 2} \frac{ln(e^{x-2})}{x^2 - 4} [3pt]$$

Solution:

Apply L'H$\widehat{o}$pital's rule: $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$

$f'(x) = 1$ since, $\frac{1(e^{x-2})}{e^{x-2}}$

$g'(x) = 2x$

$\lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{1}{2x} = 1/4$

$$\textbf{(d)} \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} [3pt]$$

Solution:

Multiply and divide by $(\sqrt{x}+3)$: $\lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$

$\lim_{x\to 9} \frac{(x-9)}{(x-9)(\sqrt{x}+3)}$

$\lim_{x\to 9} \frac{1}{(\sqrt{x}+3)} = 1/6$

Question 3

(a)Find the solution for the quadratic expression [3pt]:

$$x^2 - 8x = 1$$

Solution:

$x^2 - 8x - 1 = 0$

$a = 1; b = -8; c = -1$

$Using:$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x_1 = 8.12$

$x_2 = -0.12$

(b) Solve the system of equations [5pt]:

$$\begin{aligned} q - 2m + 3z &= 10 \\ 2q + m + z &= 7.5 \\ -3q + 2m - 2z &= -20 \end{aligned}$$

Solution:

Multiply the second equation by 2:

$4q + 2m + 2z = 15$

Add equations 1+2 and 1+3:

$5q + 5z = 25$

$-2q + z = -10$

Multiply the 3rd equation by (-5):

$10q - 5z = 50$

Sum the second and third equations to find q:

$15q = 75$

$q = 5$

Plug q=5 into the second equation:

$5(5) + 5z = 25$

$z = 0$

Replace q and z into the first equation to find m:

$m = -2.5$

(C) Find the derivative [4pt]:

$$\frac{\partial}{\partial x}[\sqrt{4x^5 - x^2}]$$

Solution:

$\frac{\partial}{\partial x}[\sqrt{4x^5 - x^2}] = \frac{1}{2} * (4x^5 - x^2)^{-1/2} * (20x^4 - 2x)$

Question 4

Calculate the following:

(a) $\int_2^4 x^2 \mathrm{d}x[3pt]$

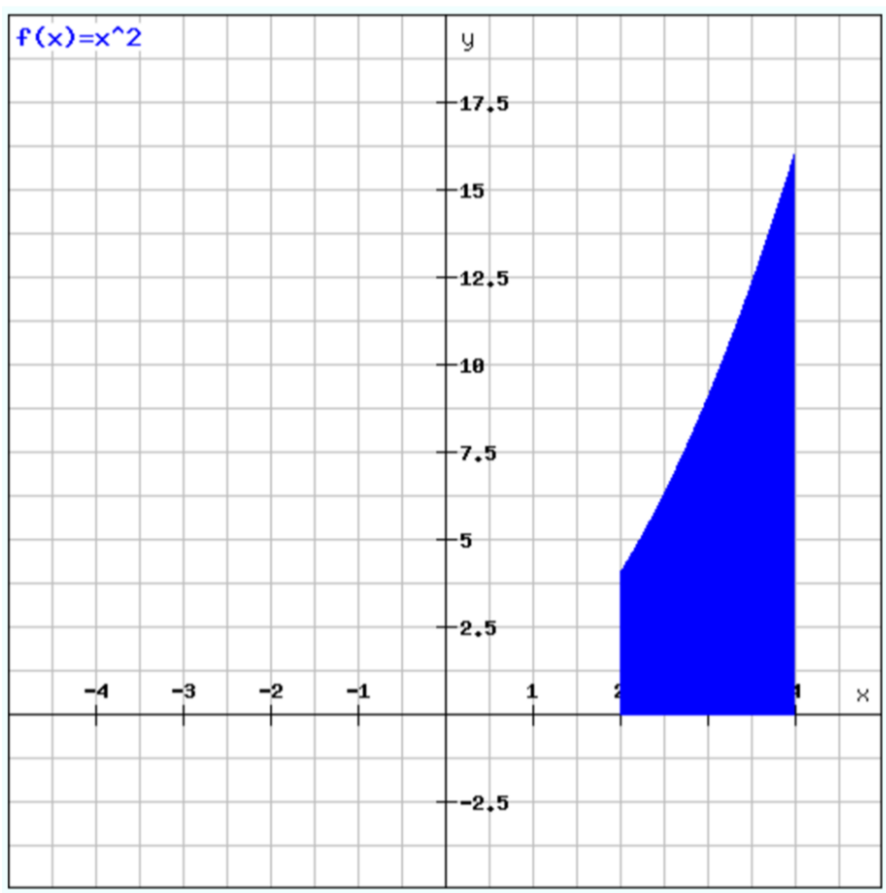
Solution

$\left[\frac{x^3}{3}\right]_{x=2}^{x=4}$

$\frac{1}{3} * (64) - \frac{1}{3} * (8) = 56/3$

(b) Graph the integral (area under the function) in part (a) on the X,Y plane. [5pt]

Solution:



(c) $\int_{-6}^{6}(13x^3 + 10\sqrt{x} + 12)\mathrm{d}x[4pt]$

Solution

$\left[\frac{13x^4}{4} + \frac{20x^{3/2}}{3} + 12x\right]_{x=-6}^{x=6}$

$144 + (40 + 40i)\sqrt{6})$

Question 5

A box manufacturer, hires you to help them figure out the optimal cost of their new product. The product is a rectangular container with an open top that requires a volume capacity of $20m^3$.

The length of the base is twice the width of the box and the materials for the base cost 20 pounds per square meter. The materials for the sides cost 12 pounds per square meter.

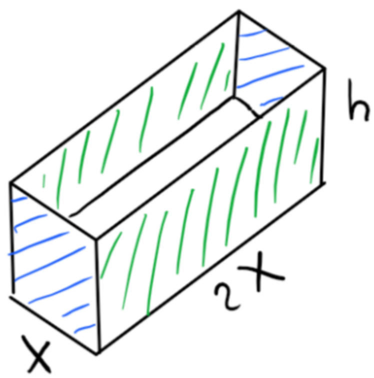


Figure 1: Open Box

a) Find the cost function as a function of (x) [7pt]

Solution:

Cost = Cost of the base + Cost of the sides

$C(x) = 20 * (x2x) + 2(12xh) + 2(12(2xh))$

$C(x) = 40x^2 + 72xh$

Volume $= 20m^3 = 2x^2h$

$h = \frac{10}{x^2}$

$C(x) = 40x^2 + 72x(\frac{10}{x^2})$

$C(x) = 40x^2 + 720x^{-1}$

b) Find if the function has a minimum or a maximum [3pts]

$C'(x) = 80x - 720x^{-2}$ $\forall x$, except $x = 0$

$C'(x) = 80x - 720x^{-2} = 0$

$x = \sqrt[3]{9} = 2.08$

Now take the second derivative to determine if it is a min or a max:

$C''(x) = 80 + 1440x^{-3} > 0$

The function is convex and $x = 2.08$ is a minimum.

c) Find the optimal cost of materials for the new product development [5pts]

$C(2.08) = 40(2.08)^2 + \frac{720}{2.08}$

$C(2.08) \cong 519.2$

Question 6

Find: $\int\int_D (4xy - y^3)\mathrm{d}A$**, D is the region bounded by the functions** $y = \sqrt{x}$ **and** $y = x^2$ **[10pt]:**

Solution:

First from the graph of the functions it is possible to show that:

$0 < x < 1$ and $x^2 < y < \sqrt{x}$

We can do the integral:

$\int_0^1 \int_{x^2}^{\sqrt{x}} (4xy - y^3)\mathrm{d}y\mathrm{d}x$

$\int_0^1 (2xy^2 - \frac{1}{4}y^4)\Big|_{x^2}^{\sqrt{x}} \mathrm{d}x$

$\int_0^1 (\frac{7}{4}x^2 - 2x^5 + \frac{1}{4}x^8)\mathrm{d}x$

$\frac{7}{12}x^3 - \frac{2}{6}x^6 + \frac{1}{36}x^9 \Big|_0^1 = \frac{10}{36}$

Question 7

Consider the following matrix A:

A = $\begin{bmatrix} 7 & 5 \\ 6 & 14 \end{bmatrix}$

Find: A^{-1} **[5pt];** $det(A^{-1})$ **[2pt];** $A^{-1} * I_2$ **[3pt], where** I_2 **is the identity matrix of size** $[2 \times 2]$**.**

Solution:

$A^{-1} = 1/(68) \times \begin{bmatrix} 14 & -5 \\ -6 & 7 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} \frac{7}{34} & \frac{-5}{68} \\ \frac{-3}{34} & \frac{7}{68} \end{bmatrix}$

$det(A^{-1}) = \frac{1}{68}$

$A^{-1} * I_2 = \begin{bmatrix} \frac{7}{34} & \frac{-5}{68} \\ \frac{-3}{34} & \frac{7}{68} \end{bmatrix}$

Question 8

You ask for a loan of 100 **pounds today. The interest rate for the loan is** r**. Find r if the total amount owed after (n=10) years is equal to** 2000 **pounds (Note: The sum was continuously compounded for n years.). [10pt]**

Solution:

$100 * e^{10*r} = 2000$

$e^{10*r} = 20$

take ln of both sides:

$ln(e^{10r}) = ln(20)$

$10r = 2.995732$

$r = 0.2995732$

Question 9

A factory in China produces smart phones according to the following revenue and cost functions:

$R(x) = 25x$

$C(x) = x^3 - 9x^2 + 35x$

Build the profit function (π) and find the optimal amount that the firm should produce in order to maximize their profit.[pt 10]

Solution:

$\pi(x) = 25x - x^3 + 9x^2 - 35x$

$\pi(x) = -x^3 + 9x^2 - 10x$

$\pi'(x) = -3x^2 + 18x - 10$

$\pi'(x) = -3x^2 + 18x - 10 = 0$

$3x^2 - 18x + 10 = 0$

$x = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(3)(10)}}{2(3)}$

$x_1 = 5.38$

$x_2 = 0.62$

Replacing in the second derivative:

$\pi''(x) = -6x + 18$

Replacing x_1 we find that

$$\pi''(x_1) < 0$$

Replacing x_2 we find that

$$\pi''(x_2) > 0$$

Then, x_1 must be a maximum. We can find the total maximum profit form producing $x = 5.38$ thousand units:

$\pi(5.38) = -(5.38)^3 + 9(5.38)^2 - 10(5.38)$

$\pi(5.38) = 50.98$