Simultaneous Search and Network Efficiency*

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Abstract

When workers send applications to vacancies they create a bipartite network. Coordination frictions arise if workers and firms only observe their own links. We show that those frictions and the wage mechanism are in general not independent. Wage mechanisms that give rise to ex ante wage dispersion are inefficient in terms of network formation and only wage mechanisms that allow for ex post competition generate the maximum matching on a realized network. Finally, we provide a decentralized wage mechanism that implements the social planner's solution and generates the maximum expected number of matches given market tightness and search intensity.

Keywords: Efficiency, network clearing, random network formation, simultaneous search.

JEL-Classifications: D83, D85, E24, J64

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1 Introduction

When workers apply to one or more jobs, a network arises where each application establishes a link between a worker and a firm. In such a decentralized environment there are two coordination frictions, (i) workers do not know where other workers apply to and (ii) firms do not know which workers are considered by other firms. We can think of the first coordination friction as referring to random network formation, while the second coordination friction affects network clearing (the number of matches on a given network). Treating the job search process as a matching on a bipartite graph (network) gives new insights into one of the key questions in the labor-search literature namely, under which conditions is the decentralized market outcome constrained efficient? With constrained efficiency we mean that the market outcome is identical to the outcome of a hypothetical social planner who maximizes social welfare given the fundamental frictions (i) and (ii).

The main contribution of our paper is that it shows how under directed search (workers observe the posted wages before applying), the wage mechanism affects frictions through network formation and clearing.[1] We find that efficient network formation requires that identical vacancies have the same application arrival rate (this implies no ex ante wage dispersion) and that efficient network clearing requires ex post competition between firms that consider the same candidate. The efficiency condition in Kircher (2009), where workers send multiple applications and firms can contact all workers, imposes however that some vacancies should have a higher probability to receive an application than others. The difference between our efficiency condition and Kircher's occurs because he places more restrictions on the planner's network clearing mechanism.

Wage mechanisms that allow for ex post competition generate the maximum number

[1]Coles and Eeckhout (2003) and Eeckhout and Kircher (2010) show that the number of matches in a model with identical workers is independent of the posted wage mechanism. We show that this no longer holds if workers send multiple applications. When workers apply to only one job, only the first coordination friction occurs, since all firms that receive at least one application can be sure that their selected candidate has no competing offer from another firm, see Burdett, Shi and Wright (2001). In the random search models of Diamond (1982), Mortensen (1982) and Pissarides (2000) the wage determination process and the matching process are fully independent. In Moen's (2000) competitive search model, workers can sort in submarkets which are characterized by different wage and market tightness pairs. Within each submarket, given market tightness, the number of matches does not depend on wages.

of possible matches on a realized network and are therefore socially efficient. This happens because firms can respond to a particular realization of the network by increasing their posted wages. Specifically, firms that have n candidates who are collectively linked to more than n firms will bid more aggressively than firms with n candidates who are collectively linked to less than n firms. Finally, we show how in a decentralized economy, workers and firms can reach the maximum number of matches through offers and counter offers. The mechanism only requires the agents to know their own links and not the entire network. The efficiency results for network formation and network clearing are for given labor market tightness and search intensity. At the end we briefly discuss why the decentralized economy will not be socially efficient in terms of entry and search intensity. Combining our and Kircher's (2009) results, suggests that there may not exist a decentralized wage mechanism that is efficient in all dimensions.

Our paper is the first one that analyzes how standard decentralized wage mechanisms affect network formation and network clearing in a decentralized search model with complete recall where workers only know to which firms they applied and firms only know which workers applied to them. The only other paper that we are aware of that considers a search model with multilateral negotiations where workers and firms do not know the entire network is Elliot (2011). He focuses on the efficiency of entry and search intensity and allows for heterogeneity. In his wage mechanism, the bargaining power is assumed to be independent of the type of subgraph an agent is in, while we show that the type of subgraph determines the agents' payoff.[2] Manea (2011) considers a framework where agents who are connected in a network are randomly selected to bargain. During the bargaining game they are not able to contact other connected agents. His random selection setting implies that a firm with many candidates has a stronger bargaining position, because it is more likely to be selected. In our model it is not the number of candidates that matters but whether a firm is located in a subgraph with more firms than workers.

Part of the network literature has analyzed different pricing mechanisms and has studied whether these price mechanisms lead to an efficient matching of sellers and buyers.

[2]Following Corominas-Bosch (2001) each graph can be decomposed into worker subgraphs with an excess number of workers, firm subgraphs with an excess number of firms and even subgraphs with an equal number of workers and firms.

Kranton and Minehart (2001) show for example that a public ascending price auction ensures efficient network clearing. Corominas-Bosch (2004) shows for identical sellers and buyers that an alternating-offers game where all sellers (or buyers) of a subgraph simultaneously announce prices, leads to a maximum matching. This literature, however, assumes that once a network has been formed, all agents know the complete network (or the entire subgraph of the network they are in).[3] This knowledge allows sellers and buyers to determine their exact outside option (trading partners and trading prices). We show that ex post competition achieves the maximum matching, even if agents do not know the network structure. Another part of the network literature uses the set-valued approach, i.e., it either starts with a set of competitive price vectors and shows that the resulting matches are pairwise stable and maximize aggregate welfare (see Kranton and Minehart, 2000), or it starts by assuming that pairwise stable matches must arise and then analyses wage formation (see Elliott, 2011). Those papers do not layout the game that leads to a competitive price vector or a pairwise stable matching like we do. Moreover, pairwise stable matchings are not necessarily maximum matchings (i.e., Kircher, 2009) but a maximum matching is always stable since an improvement of one agent must make another agent worse off. Finally, there is a growing number of papers that combine insights from search and network theory.[4] Those papers focus mainly on how social networks of workers can pass information of the location of jobs on to each other, which is very different from the bipartite network (between workers and firms) framework that we consider here.

The paper is organized as follows. We start in section 2 with a 3-by-3 example that illustrates our main point that wage dispersion leads to less efficient networks and ex post competition generates a maximum matching on a given network while wage commitment does not. Sections 3 and 4 consider a large labor market. In section 3 we present a wage game with multi-round offers and counter offers. Section 4 introduces some insights from graph theory to derive two important general results. First, in section 4.2 we show that our wage game that has ex post competition and complete recall gives the maximum

[3] Galeotti et al. (2010) analyse network games with limited information. However, they only consider one type of agents, i.e., they do not consider vacancies and workers or sellers and buyers in a bipartite network.

[4] Example include, Boorman (1975), Calvó-Armengol and Jackson (2004), Calvó-Armengol and Zenou (2004), Fontaine (2004).

matching on a given network and that wage mechanisms without ex-post competition do not. In section 4.4 we show that in terms of network formation, workers should apply to each vacancy with equal probability. This only occurs, if all firms post the same wage or if search is random (workers do not observe the wage ex ante). This implies that the wage game of section 3 decentralizes the Planner's solution in terms of network formation and clearing. Finally section 5 concludes.

2 A simple example

This section illustrates our main points that (i) ex ante wage dispersion leads to less efficient network formation and that (ii) ex post competition generates a maximum matching in a simple environment with 3 workers, 3 firms and 2 applications per worker.[5] Workers do not know to which firms other workers apply to. First, we look at network formation and assume that network clearing generates the maximum number of matches. Then, we look at which wage mechanisms are most efficient in terms of network clearing. Efficient network clearing implies that the number of matches is equal to 3, if each of the three vacancies receives at least one application, and equal to 2, if only two vacancies receive applications. Note that these are the only two possible outcomes, since no worker sends both applications to the same firm. Let ξ_i be the probability that a worker sends one of her two applications to vacancy i. Under the assumption that network clearing is efficient, the expected number of matches is,

$$M = \sum_{i=1}^{3} \left(1 - (1 - \xi_i)^3\right), \text{ with } \sum_{i=1}^{3} \xi_i = 2,$$

where $(1 - \xi_i)^3$ equals the probability that vacancy i does not get any application. Since the function $\left(1 - (1 - \xi_i)^3\right)$ is concave in ξ_i, Jensen's inequality implies that the number of matches is maximized, if all vacancies have the same probability to receive an application, i.e., if $\xi_i = 2/3$. Thus, only wage mechanisms that generate no ex ante wage dispersion (for example, all mechanisms under random search) can lead to the maximum number of matches, $M = 26/9 \approx 2.889$.

[5] If workers send 1 application or 3 applications, the number of matches generated is independent of the wage mechanism used.

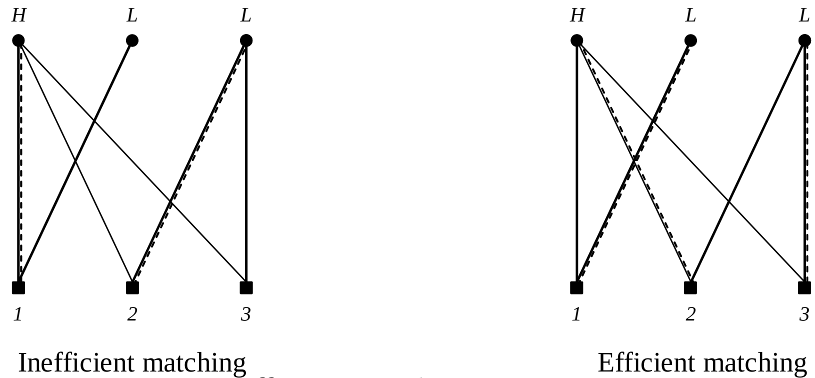

Figure 1: Inefficiency without ex post competition

Next, consider network clearing. Efficient network clearing requires that the number of matches is equal to 3, if all three vacancies are collectively linked to all three workers, and that the number of matches is equal to 2, if only two vacancies are collectively linked to all three workers. Network clearing is in general not efficient, if firms commit to their posted wages. To see this, consider the graph in Figure 1, which pictures a particular realization of the case where each worker sends one application to the high-wage firm and one to one of the two low-wage firms (thick lines).

The number of matches (dashed lines) now depends on which worker is chosen by the high-wage firm. If the high-wage firm offers the job to one of the workers who are linked to the low-wage firm with two applicants, i.e., to worker 2 or 3 in Figure 1, the number of matches is equal to the maximum number of matches (3). If the high-wage firm offers the job to the worker linked to the low-wage firm with only one applicant, i.e. to worker 1 in Figure 1, there are only two matches, since the low-wage firm with only one applicant will remain unmatched. If this firm *could* ex post increase its initial offer it would bid the high wage plus epsilon and hire worker 1 while the high-wage firm would hire one of its other candidates. It is easy to show that in this example, allowing for ex post competition always leads to the maximum number of matches. This illustrates that the wage mechanism and the matching process are not independent. Different search environments generate different distributions of networks and whether the wage mechanism allows for ex post competition or not affects the number of matches for a given network. A final important point is that in both cases, the matching is stable. Worker 2 would prefer his match with the H firm in the inefficient M=2 case rather than matching with the L-firm.

3 A wage game with complete recall and ex-post competition

Before presenting our main results for a large labor market, we first lay out the precize setting and the timing of events. Consider v identical firms with one vacancy each and u identical risk neutral unemployed workers, who can send $a \leq v$ applications to different firms. Search is random, i.e., workers send each application with probability $1/v$ to any specific vacancy. Workers have a reservation wage of 0 and a matched firm-worker pair produces 1. We restrict ourselves to symmetric and anonymous strategies. Symmetry implies that workers cannot coordinate in where to send their applications while anonymity implies that firms must treat identical workers similarly and vice versa (see Burdett, Shi and Wright, 2001). Finally, we take the number of applications that workers send out and market tightness as given. The main reason for this is that the conditions for efficient entry and the number of applications are well known and have been studied before.[6] This allows us to focus on the efficiency of random network formation and network clearing. However, at the end we will make some remarks on the efficiency of entry and search intensity.

3.1 Network clearing (assignment game)

We start with the network-clearing or assignment game. The realized network that is formed by the random application process is unknown to workers and firms. The following timing of the assignment game also describes the action and information sets of workers and firms:

1. Each firm selects one worker (if present) and offers that worker a wage $w \geq 0$. Wage offers are discrete $w \in \{\Delta, 2\Delta, ..., 1-\Delta, 1\}$, where Δ is a small but discrete amount, i.e., a cent.

[6] Gautier and Moraga-Gonzalez (2005) and Albrecht et al. (2006) find without recall, that workers send too many applications (due to rent seeking and congestion externalities) and that entry is excessive, because firms have too much market power. Kircher (2009) shows that with directed search, wage commitment and full recall, entry and search intensity are socially efficient. Elliot (2011) finds that firm entry is never excessive but can be too small and that workers send too many applications.

2. A worker with one or more offers can keep at most one offer (which is observed and verifiable by all linked firms) and must reject all others. The worker and the only not rejected firm are labeled to be *engaged*.

3. The firms that are rejected select a worker (possibly the same worker) and offer that worker a wage $w \geq 0$ given the wage offers from other firms that are kept by their applicants.

4. The engaged firms can make a counter offer.

5. A worker with one or more offers can keep at most one offer (observed by all linked firms) and must reject all others.

6. Return to stage 3)... until the final round T (sufficiently large).

We assume the following tie breaking rule for workers. Workers keep the offer of the engaged firm if it offers the same wage as the highest offer made by any other firm. If the worker was not engaged and two or more outside firms offer the same highest wage, the worker randomly picks one of them and rejects the others.

The network clearing game assumes that firms have all the bargaining power and can make take-it-or-leave-it wage offers. An interesting alternative is the case where workers have all the bargaining power. Kim and Kircher (2012) show that from a welfare point of view this is more desirable. In that case, the network clearing game is the same as above except that we have to exchange the roles of firms and workers (workers now make a take-it-or leave-it offer, $w \leq 1$).

3.2 Network decomposition

In order to determine the optimal strategies for workers and firms we use the properties of the Decomposition Theorem by Corominas-Bosch (2004) (for details see Appendix B.1), which – in terms of our terminology – decomposes a realized network into firm-, worker- and even subgraphs. A firm subgraph contains more firms than workers. A worker subgraph contains more workers than firms. In even subgraphs, the number of workers equals the number of firms (see Figure 2).The algorithm first looks for firm subgraphs and separates all of them from the network. Then it identifies worker subgraphs and removes all of them from the network. The remaining subgraphs are even subgraphs. The decomposition is not unique but the Decomposition Theorem states that any firm and any worker will always belong to the same type of subgraph, a property important to guarantee that the different possible decompositions are payoff equivalent.

Figure 2 illustrates the Decomposition Theorem. The algorithm starts with the first firm and identifies a set of firms as firm subgraph if it has less neighbors (more precisely, if it is jointly linked to less neighbors, i.e., $|F| > |N(F)|$). In order to ensure that the maximum matching is found, the algorithm has to start with $|F| = 1$. The number $|F|$ increases by one once all firm combinations with $|F|$ have been considered (Hall's Theorem, 1935). The first subgraph in Figure 2 is the unmatched firm G. The firm subgraph G_1^f is removed before the algorithm continues. Since there are no firm subgraphs with $|F| = 2$, the next firm subgraph has three firms, i.e., $|F| = 3$, The three firms A, B and C in this subgraph are collectively linked to workers 1 and 2, i.e., $N(\{A, B, C\}) = \{1, 2\}$ and $|N(\{A, B, C\})| = 2$. Once the firm-subgraph G_2^f is removed, it is easy to identify that the remaining sets of firms are collectively linked to more neighbors, i.e., $|F| \leq |N(F)|$. Hence, there are no further firm subgraphs. The algorithm continues by looking for worker subgraphs in the same way as it looked for firm subgraphs. At $|W| = 4$, the algorithm identifies a worker subgraph with $N(\{3, 4, 5, 6\}) = \{D, E, F\}$ and $|N(\{3, 4, 5, 6\})| = 3$. Once the worker subgraph G_1^w is removed, and no further worker subgraphs are found the algorithm stops by identifying all remaining subgraphs as even subgraphs, i.e., in Figure 2 the remaining subgraph G_1^e is an even subgraph with $N(\{7, 8\}) = \{H, I\}$ and $|N(\{7, 8\})| = 2 = |\{H, I\}|$.

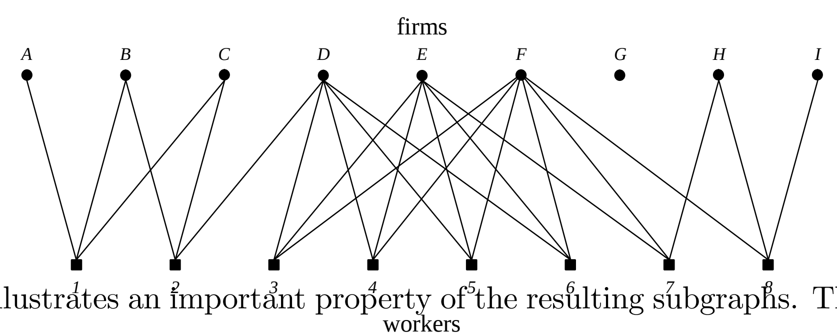

Figure 2 illustrates an important property of the resulting subgraphs. The long side of a subgraph has only links to the short side of the respective subgraph, i.e., the workers in a worker-subgraph, G_i^w (firms in a firm-subgraph, G_i^f) are only linked to firms (workers)

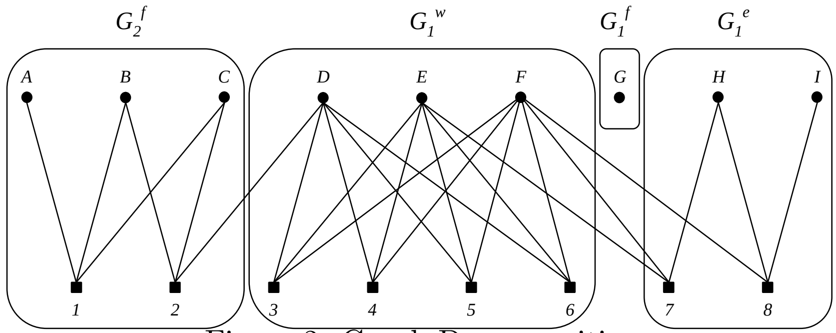

Figure 2: Graph-Decomposition

in a worker (firm) subgraph. Workers in an even subgraph, G_i^e, are only linked to firms in worker- or even subgraphs and the firms in even subgraphs can only be linked to workers in firm- or even subgraphs.

3.3 Information sets and beliefs

The actions of firms (or more general, the agents who have the power to propose the wage) will depend on their belief about the subgraph they are in. Firms will update their beliefs given the number of applicants N they have and the wage offers of their applicants, i.e., the set of wage offers W^N the applicants hold from their engaged firms. Denote the belief of firm j in round t that it is in a firm-subgraph given N and W^N, by $b_{j,t}\left(N, W^N\right)$. I.e., if a firm is sure to be in a firm subgraph $b_{j,t}\left(N, W^N\right) = 1$. Firms without any applicant are by definition in a firm-subgraph (see G_1^f in Figure 2). Firms with at least one applicant can be in any type of subgraph.

Let us have a closer look at the information set of firms. We will show below that firms can infer from the (sub)sets of observed wage offers W^N whether they are in a firm-subgraph or not. Let us therefore define the following subsets. If $k \in \{0, 1, ..., n\}$ applicants receive no offer, denote the respective subset of wage offers by Ø^k. If $l \in \{0, 1, ..., n\}$ applicants hold wage offers equal to zero, denote the respective subset by 0^l. If $m \in \{0, 1, ..., n\}$ applicants hold an offer equal to one cent, denote the respective subset by Δ^m. Finally, if $q \in \{0, 1, ..., n\}$ applicants hold an offer equal to one, denote the respective subset by 1^q. Thus, the set of wage offers is equal to $W^N = \left\{\widetilde{W}^{N-k-l-m-n}, \text{Ø}^k, 0^l, \Delta^m, 1^q\right\}$, where $\widetilde{W}^{N-k-l-m-q}$ equals the remaining subset of wage offers with wages $w \in \{2\Delta, 3\Delta, ..., 1-\Delta\}$. We show below that in equilibrium $\widetilde{W}^{N-k-l-m-n}$

is empty in the final round given that T is sufficiently large.

3.4 Workers' and firms' strategies

Below, we prove that the following set of strategies constitutes a perfect Bayesian Nash equilibrium to the assignment game.

Consider the following worker strategies:

A1 In the final round T, accept the best offer.

A2 In any previous round $t < T$, keep the best offer $w^h = \max\{w^1, w^2, ..., w^a\}$, and reject all other offers. Accept the best offer, if $w^h = 1$.

Engaged firms have one advantage over rejected firms. They can make a counter offer before the next round starts. This implies that they can base their actions on the set of wage offers, W^N, they observe and do not need to base their actions on the beliefs about the type of subgraph they are in. We therefore consider the following counter offer strategies for engaged firms:

B1 In the final round T, match any offer.

B2 In any previous round $t < T$, match any outside offer $w^h = 0$. Match the offer $w^h \geq \Delta$, if all other applicants hold an offer $\widetilde{w}^h \geq w^h - \Delta$, and don't match the offer $w^h \geq \Delta$, if at least one other applicant holds no offer or an offer $\widetilde{w}^h < w^h - \Delta$.

For rejected firms in rounds $t < T$ we consider strategies that are independent of the firm's belief $b_{j,t}\left(N, W^N\right)$ and only in the final round T we consider strategies that depend on the belief $b_{j,T}\left(N, W^N\right)$. The reason is that in any round $t < T$ rejected firms have the option to increase their offer in the next round.

C1 In the final round T, the strategy is as follows:

C1a If at least one applicant holds no offer, i.e., $W^N = \left\{W^{N-k-l-m-q}, \varnothing^k, 0^l, \Delta^m, 1^q\right\}$ with $k > 0$, then offer one of the k applicants $w = 0$ if $b_{j,T}\left(N, W^N\right) = 0$, else offer one applicant $w \in F(w)$ if $b_{j,T}\left(N, W^N\right) \neq 0$, where the optimal wage offer distribution $F(w)$ is characterized in Gautier and Moraga-Gonzalez (2004). (Note, that we show below that for T sufficiently large $b_{j,T}\left(N, W^N\right) = 0$ if $k > 0$.)

C1b If all applicants hold an offer, select one worker and offer him $w = 1$ (irrespective of $b_{j,T}\left(N, W^N\right)$).

C2 In any previous round $t < T$, the strategy is as follows:

C2a If at least one applicant holds no offer, i.e., $W^N = \left\{W^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q\right\}$ with $k > 0$, offer one of the k applicants $w = 0$ irrespective of the belief $b_{j,t}\left(N, W^N\right)$.

C2b If all applicants hold an offer $w^h \geq 0$, i.e., $W^N = \left\{W^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q\right\}$ with $k = 0$, offer the worker with the lowest best offer $\underline{w}^h = \min W^N$ the wage $w = \underline{w}^h + \Delta$ if $\underline{w}^h < 1$ irrespective of the belief $b_{j,t}\left(N, W^N\right)$ and $w = 1$ if $\underline{w}^h = 1$.

3.5 Wages and beliefs

In order to show that the proposed strategies are indeed optimal it will be useful to analyze first the wages that are paid in each type of subgraph.

Lemma 1 *Workers' and firms' strategies imply that*
(i) at $t \geq u$ all firms in worker subgraphs are engaged and their engaged workers hold an offer no higher than $w = 0$,
(ii) at $t \geq 2u$ all workers and firms in even subgraphs are engaged and all workers hold an offer $w \in \{0, \Delta\}$,
(iii) at $t \geq u/\Delta$ all workers in firm subgraphs are engaged and hold an offer $w = 1$.

Proof. See Appendix A.1. Lemma 1 shows that after $t \geq u/\Delta$ rounds have passed, all workers in firm subgraphs will have received an offer $w^h = 1$ and have accepted it such that all other firms that are not part of a firm subgraph can infer from the absence of wage offers $w^h = 1$, i.e., from $q = 0$ at $t \geq u/\Delta$, that they are not in a firm subgraph. This is stated in part (ii) in the following Lemma.

Lemma 2 *(i) Rejected firms that observe that their applicants hold wage offers* $W^N = \left\{W^{N-m-n}, \Delta^m, 1^n\right\}$ *with* $k = l = 0$ *in round* $t \geq 2u$ *hold a belief*

$$b_{j,t}\left(N, \left\{W^{N-m-n}, \Delta^m, 1^n\right\}\right) = 1.$$

(ii) At $t = u/\Delta$ *all firms that observe* $W^N = \left\{\varnothing^k, 0^{N-k-m}, \Delta^m\right\}$ *hold a belief*

$$b_{j,u/\Delta}\left(N,\ \left\{\varnothing^k, 0^{N-k-m},\ \Delta^m\right\}\right) = 0.$$

Proof. See Appendix A.2

For other values of $W^N\{\cdot\}$ beliefs can be between 0 and 1. However, actions only depend on beliefs in the final round so we do not care about them.

3.6 Equilibrium of the assignment game

Proposition 1 *The strategy profile A1-A2, B1-B2 and C1a-C2b constitute a perfect Bayesian Nash equilibrium to the assignment game.*

Proof. See Appendix A.3

We show that for workers it is optimal to always keep the best offer (since it may not be available in later rounds). The option for engaged firms to make counter offers is crucial to rule out strategic behavior in order to manipulate beliefs. Suppose for example there are 3 firms (A,B,C) in a firm graph with 2 workers (1,2) who applied to all three firms. Suppose only firm A has learnt that it is in a firm graph. B and C will continue to believe they are in an even graph and offer $w = 0$ to workers 1 and 2 as long as their workers do not show better offers. Why is it then not in the interest of firm A to make no offer till $T - 1$ and then offer Δ in round T? This is not profitable because in that case B or C will match this offer as long as firm A offers less than 1. If A would offer 1 in the last round it may as well immediately make an offer Δ in the second round.

4 General results on random network formation and network clearing

The example of section 2 suggests that ex ante wage dispersion is inefficient in terms of random network formation and that we need ex post competition in order to get efficient network clearing. In this section we use some results from (random) graph theory to show that those results hold in general. In section 4.2 we show that maximum matching requires ex post competition and in section 4.4 we show that all firms post a wage equal to the reservation wage, which leads to efficient network formation by ensuring that all vacancies have the same application arrival rate.

4.1 Berge's Theorem

We first briefly describe some basic concepts of graph theory that are relevant for our environment. When workers apply to jobs, each of their applications is a link (or edge) in a bipartite network (or graph). The wage mechanism and search environment determine both the distribution of networks that can arise and the matching on a given network. The networks (or graphs) in our environment are simple (workers do not send multiple applications to the same firm), undirected (if worker i is linked to firm j, then firm j is linked to worker i) and bipartite ($G = \langle u \cup v, L \rangle$ consists of a set of nodes formed by two different kind of agents, i.e., by workers and vacancies, and a set of links L where each link connects a worker to a firm, so workers are not linked to other workers and firms are not linked to other firms).

Definition 1 *A matching M in a graph G is a set of links such that every node of G is in at most one link of M.*

Central to our result that a maximum matching requires ex-post competition is the following theorem by Berge.

Berge's Theorem (1957):

A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

In our bipartite graph environment an M-augmenting path is defined as a path where

1. worker-firm links that are part of the matching M alternate with worker-firm links that are not part of the matching M (definition of an M-alternating path) and

2. neither the origin (firm or worker) nor the terminus (worker or firm) of the path is part of the matching M.

Figure 3 depicts an M-alternating path and an M-augmenting path in a particular network. The dots represent vacancies and the squares unemployed workers. The solid lines represent applications ($a = 2$) and the dashed lines represent matched worker-firm pairs. The M-alternating path in the first panel ($A - 1 - B - 2 - C - 4$) starts with the matched vacancy A and ends at the matched worker 4. The M-augmenting path ($A - 1 - B - 2 - C - 4$) in the second panel of Figure 3 starts with an unmatched vacancy, A, and ends with an unmatched worker, 4.

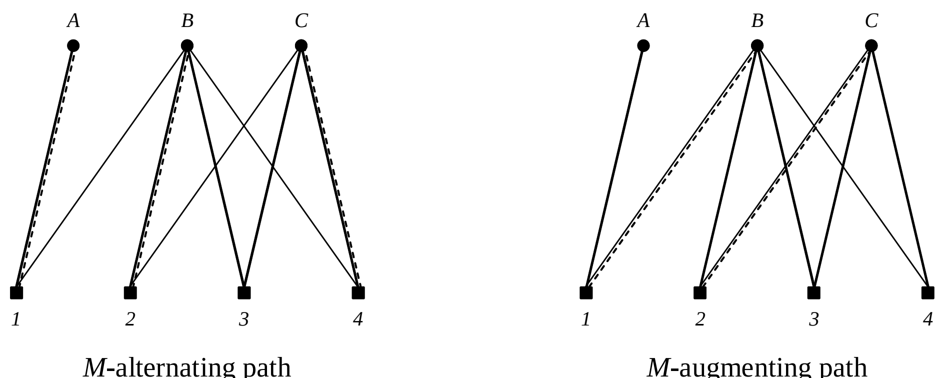

Figure 3: M-alternating path and M-augmenting path

Berge's Theorem, translated to our setting, implies that a maximum matching in a graph is only guaranteed, if an unmatched firm is not linked to an unmatched worker via an M-augmenting path. The reason that a matching is not optimal, if an M-augmenting path exists, is that one could create one more match by switching the links. Then, the unmatched firm at the start of the M-augmenting path and the unmatched worker at the end of the M-augmenting path will both be matched at the expense of one match in the middle. Comparing the two paths in the second panel of Figure 3 illustrates this. The matching $M = \{1 - B, 2 - C\}$ in an M-augmenting path can always be increased by *switching* the dashed and solid links resulting in an extra link, i.e., $M = \{A - 1, B - 2, C - 4\}$.

What remains to be shown is that if a matching M has no M-augmenting paths, it is a maximum matching. This can be proven by contradiction. Suppose that in a particular graph in our setting there is a matching M for which there are no M-augmenting paths but that (contrary to Berge's Theorem) this matching is not a maximum matching. Then there is a matching N (i.e. $A-1, B-2, C-4$; dashed lines in Figure 4) with more links than M (i.e. $1-B, 2-C$; dotted lines in Figure 4), $|N| > |M|$. Now consider the symmetric difference $N \Delta M$ defined as the set of links that is either in N or M but not in both (the sum of dashed and dotted lines in Figure 4, $A-1$, $B-2$, $C-4$, $1-B$, $2-C$). Each worker or firm can have at most 2 links in $N \Delta M$ because he is hired by at most one firm in M and at most one firm in N. Moreover, the links of the paths alternate between being in M and being in N, because by the definition of a matching, no node can have two links in M or two links in N. Since by assumption N is strictly bigger than M there must be at least one path in $N \Delta M$ with an odd number of links that starts with a firm (worker) in N and ends with a worker (firm) in N (i.e., $A-1-B-2-C-4$). But then this is an M-augmenting path because the firm and worker at the start and end of the path are (by the symmetric difference operation) not in M. This gives us the desired contradiction, because we started by assuming that M has no M-augmenting paths.

Thus, in order to show that ex-post competition leads to a maximum matching we need to rule out that an M-augmenting path exists.

4.2 Maximum matching requires ex post competition

In this section we show that for a given network, ex post competition with complete recall generates a maximum matching.

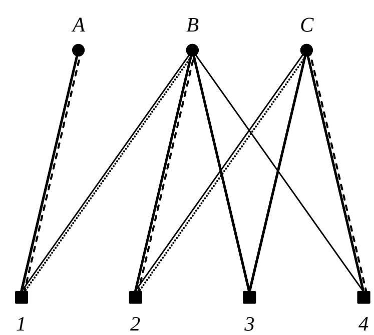

Figure 4: Symmetric difference operation ($N \Delta M$)

4.3 The ex-post competition game gives a maximum matching

Next, we prove in four simple steps that the network-clearing subgame generates a maximum matching on any given network. Let x be the smallest monetary unit, i.e. a matching offer must be an amount x higher than the previous one.

Lemma 3 *If a firm remains unmatched after T rounds, then all workers along an M-alternating path that starts with the unmatched firm must earn a wage equal to the marginal product, i.e., $w = 1$.*

Proof: To see why all workers along the M-alternating path receive $w = 1$, first note that if a firm with candidates (firm A) remains unmatched after T rounds, then all its applicants must have accepted a wage $w = 1$ (since if at least one of its candidates would earn $w < 1$, firm A would have offered that worker $w < 1$ and make positive profits). But then the other candidate of the next firm along the M-alternating path (firm B) that hired A's candidate must also receive $w = 1$ otherwise firm B would have hired that worker at a $w < 1$. Repeating this argument implies that all firms along the M-alternating path pay a wage of 1. ■

Lemma 3 implies that all workers in M-alternating paths that start with an unmatched firm have been offered a wage equal to 1 and have left the market after $T - 1$ rounds.

Lemma 4 *If a worker remains unmatched after T rounds, each firm along an M-alternating path that starts with the unmatched worker pays no more than the highest posted wage.*

Proof: The firm (firm A) to which the unmatched worker (worker 1) applied will offer the worker who it hired (worker 2) at most its posted wage otherwise it could have offered the job to the unmatched worker 1. But then the worker (worker 3) who is hired by the next firm along the M-alternating path (firm B) must also earn weakly less than the highest posted wage, else his firm (B) would have hired worker 2. Repeating this argument implies that all firms along the M-alternating path that starts with an unmatched worker pay a wage less than the highest posted wage. ■

Lemma 4 implies that all workers in M-alternating paths that start with an unmatched worker have been offered a wage no higher than the highest posted wage after $T-1$ rounds.

Lemma 3 above shows that this holds for all M-alternating paths that do not start with an unmatched firm. It follows that all workers who requested a wage above the highest posted wage have left the market after $T-1$ rounds.

Lemma 5 *The highest posted wage is strictly smaller than* 1*.*

Proof: Under directed search, any firm that offers a wage equal to 1 makes no profit and could increase its profits by offering a wage strictly less than one since there is a positive probability that one of its candidates receives no better offers and accepts this lower offer in round T. ■

According to Berge's Theorem a maximum matching exists if and only if there is no M-alternating path that starts with an unmatched worker and ends with an unmatched firm, i.e., if and only if there is no M-augmented path. Given the wage pattern in an M-alternating path that starts with an unmatched worker (Lemma 2) or with an unmatched firm (Lemma 3), we can write down our main Proposition.

Proposition 2 *Ex-post competition leads to a maximum matching in any realized network.*

Proof: Suppose it would not lead to a maximum matching. In that case there would exist an M-augmenting path with at least one unmatched worker and one unmatched firm. But then Lemma 1, 2 and 3 imply that all firms along the M-augmenting path (that is also an M-alternating path) offer both a wage less than 1 and a wage equal to 1, which is a contradiction. ■

Note that this result is very general. If firms can only interview a subset of their workers as in Wolthoff (2011) or one as in Albrecht et al. (2006) and Galenianos and Kircher (2009), the realized network will be different but Proposition 2 still holds. The same is true, if workers have for example different search costs and consequently send out different numbers of applications. Also, if firms can create shortlists of at most n candidates, our result holds. This just requires an intermediate step where all firms with more than n candidates must eliminate (at random) a number of links. After this intermediate step, a new network arises for which the same results on maximum matching hold as above.

The flexibility to adjust wages ex post is central to achieve efficiency in network clearing. If firms commit to their posted wages and do not adjust their wages ex post, we can typically observe different wages along an M-alternating path. If both end nodes of the M-alternating path are unmatched, i.e., if we have an M-augmenting path, there is no mechanism inherent in the matching process associated with wage commitment that can induce the matched firm-worker pairs to rematch with the unmatched firm and worker at the end of the M-augmenting path. Thus, the inefficient network clearing result of wage commitment from the 3 by 3 example of section 2 holds in general. Therefore, Berge's Theorem also implies the following Corollary,

Corollary 1 *If firms commit not to increase their posted wages ex-post, network clearing is generally inefficient and the maximum matching is not realized.*

Corollary 1 shows that directed search models with fixed posted wages are not able to solve the second coordination friction (firms do not know which workers are considered by other firms). Thus, although directed search with fixed posted wages is constraint efficient in terms of firm entry and number of applications that workers send, see Kircher (2009), it generally does not generate the maximum matching that is possible given the network that is formed between firms and their applicants.

Proposition 2 also implies that a social planner would never want to give one subgroup of firms the right to match first. Such a property arises, if some firms offer higher wages than others and wages cannot be raised ex-post as in Kircher (2009).

Corollary 2 *It is socially inefficient to have a subgroup of firms that matches first.*

Corollary 2 implies that it is socially inefficient to have a subgroup of high wage firms that match first and a subgroup of low wage firms that match only if their candidate(s) receive no offers at a high wage firm.[7]

4.4 Efficient network formation given efficient network clearing

In our setting, network formation is random. The symmetry and anonymity assumptions do not allow workers to identify certain firms and to condition their application decision

[7]Note, that Kircher's (2009) equilibrium is constrained efficient because the planner takes the existence of a subset of firms that match first as given, whereas here this is not part of the planner's constraint.

on firms' names. Workers do observe the posted wage, and can condition their application decision on that.

The game of section 3 implies an urn-ball model of network formation, (see Albrecht, Gautier and Vroman, 2004) where workers randomly send out a applications to different firms.[8] Each application can be thought of as creating a link in a bipartite graph. This process differs from the seminal Erdös and Rényi (1960) random network formation model where each link is formed with a certain probability and the number of applications that a worker sends is a random variable.[9] In our framework the number of applications that each worker sends is given and the randomness comes from the fact that workers do not know where other workers apply. The number of applications that a firm receives is therefore a random variable. Under directed search, the expected number of applications a firm receives will of course depend on the wage (or more generally on the wage mechanism) it posts. If we make the labor market large by letting N be an arbitrary large finite number and $v \to N$ with $v/u = \theta$, the number of applications are approximately distributed according to a Poisson distribution with mean a/θ.

4.4.1 Social planner's problem

An unconstrained social planner will trivially assign each unemployed worker to a vacancy such that the number of matches equals the short side of the market. If workers send out multiple applications, the same first best assignment can be achieved, if the social planner partitions the labor market into submarkets where the number of firms and workers in each submarket is no higher than the number of applications. However, if the social planner faces the same coordination frictions as the market, he must assign symmetric strategies to identical workers, implying that he can only decide on the probability with which a worker has to send an application to a subgroup of firms.

We constrain the social planner to choose the set of firm subgroups C (where each subgroup c is defined by a certain color), the measure of vacancies v_c within each subgroup c and the probability $p_{c,i}$ that a worker sends its i-th application to subgroup $c \in C$. The

[8] See also Kircher (2009) and Galeanos and Kircher (2009) and Fontaine (2004).

[9] See Bollobas (2001) for a bipartite version.

expected number of applications sent to subgroup c is equal to

$$a_c = u \sum\nolimits_{i=1}^{a} p_{c,i}.$$

The total number of workers that apply to subgroup c equals $u_c = \left(1 - \prod_{i=1}^{a} (1 - p_{c,i})\right) u$, where $\prod_{i=1}^{a} (1 - p_{c,i})$ is equal to the probability that a given unemployed worker does not send any application to subgroup c. While vacancies can by definition only be part of one subgroup, workers can be linked to at most a different subgroups depending on where they send their applications to. Workers are, however, only part of one subgraph (worker, firm or even). Subgraphs can, therefore, contain vacancies of different subgroups, if the workers that belong to that subgraph are linked to vacancies in different subgroups.

■

In a companion paper Gautier and Holzner (2012), we show that all vacancies should have the same probability to be contacted by a worker. This makes the network as balanced as possible and therefore minimizes the fraction of firms that are not matched. Shimer (2005) derives a similar condition for a directed search environment where workers can apply to only one firm. In the setting by Galenianos and Kircher (2009), where workers can send more than one application but firms can contact only one worker, the total number of matches is also maximized, if all firms have the same probability to be contacted by a worker. In contrast, the efficiency condition in Kircher (2009), where workers send multiple applications and firms can contact all workers, differs from our efficiency condition, because he constrains the social planner to let a subgroup of firms always match first (i.e., be in a high location). Those firms in a high location should be more likely to be contacted by a worker, since this reduces the probability that a worker is not available for hiring at a firm in a low location (where firms can only match, if their candidates do not have an offer from a firm in a high location). Allowing the social planner to also choose the network clearing mechanism, Corollary 2 shows that it is not optimal to let a subgroup of firms match first. Thus, Kircher's (2009) efficiency result differs from our efficiency result, because he restricts the social planner to use a network clearing mechanism that does not allow for ex post competition.

The fact that under random search, all firms post the same initial wage establishes our main result that our decentralized game is efficient in terms of network formation and clearing. The first coordination friction that workers do not know where other workers apply is minimized in the decentralized economy because workers fully randomize.[10] In addition, ex post competition generates the maximum matching. It therefore eliminates the second coordination friction between firms, i.e., the friction that firms do not know which other workers are considered by other firms.

Finally, we can show that wage mechanisms that generate ex ante wage dispersion like Kircher (2009) are not efficient in terms of network formation.

Corollary 3 *If equally productive firms post different wages, network formation is not efficient.*

Proof. See Appendix A.4.

The intuition is simple. Wage dispersion implies that a subset of firms have a higher expected arrival rate of applicants. This creates unbalanced networks and leads to inefficient network formation.

5 Final remarks

This paper contributes to one of the fundamental question in economics namely under which conditions do decentralized markets generate constraint efficient outcomes. Our focus is on the labor market where it is common that unemployed workers simultaneously send multiple applications. This creates a bipartite network between workers and firms. In such an environment there are two coordination frictions, (i) workers do not know where other workers apply to and (ii) firms do not know which workers are considered by other firms. We show that the second coordination friction between firms can be eliminated if wages in the decentralized market are determined by ex post competition and if firms can go back and forth between their applicants. In that case, the number of matches on a given network equals the maximum number of possible matches. The first coordination friction is minimized if the decentralized market ensures that workers apply to each vacancy with equal probability. This implies that an equilibrium with ex ante wage

[10] It is hard to derive the directed search equilibrium of this game but it may very well be possible that all firms post the same wage in that case.

dispersion is inefficient in terms of network formation. We show that in a directed search equilibrium with complete recall, all firms post a wage equal to the reservation wage, which implies that the decentralized market equilibrium is equal to the social planner's solution.

An interesting question is what happens if firms can commit to richer contracts, do we get a maximum matching in that case? We believe that this requires that firms must be able to post wages conditional on each possible network realization. Given the huge number of possibilities, this is impossible in practice.

Although our wage mechanism is efficient in terms of network formation and network clearing, it may not be efficient in other dimensions like vacancy creation and search intensity. Kircher (2009) shows for example that wage commitment without ex post competition implies wage dispersion and that the resulting equilibrium is efficient in terms of search intensity and firm entry. Combining those results suggests that there may not exist a simple wage mechanism that by itself generates the constrained efficient outcome along all dimensions. It is unlikely that workers send the socially desirable number of applications because there are many externalities. For example, since workers' payoffs are independent of whether they are in a worker or even subgraph, they do not take into account that an additional application might turn a worker subgraph into an even subgraph and thereby generate an additional match. There is also a rent seeking externality caused by the fact that an additional application can increase the chances of turning an even subgraph into a firm subgraph, which then implies that the surplus goes to the workers and no longer to firms. The probabilities that these events occur and consequently the expected payoffs will depend on market tightness and the aggregate search intensity. Depending on which one is more likely, workers either send too many or too few applications. Kim and Kircher (2012) show that those externalities disappear if workers awarded the worker-payoff-maximizing point in the core. Finally, firms fail to take into account that when they enter they also destroy the expected payoffs of other firms by making it more likely that these firms end up in firm subgraphs.[11] It is not clear whether the Kim-Kircher

[11] Consider for example the case where there are 10 unemployed workers sending out 10 applications to 10 vacancies. The social contribution of an additional firm is zero but the private contribution is positive, because there is a positive probability that the entrant ends up in an even subgraph.

(2012) mechanism internalizes this because they do not allow for multiple applications. We plan to investigate this in future work.

Another important and interesting extension for future research is to allow for heterogeneity in firm and or worker types, see Shimer (2005) and Elliot (2011). We conjecture that this makes ex post competition equally desirable as in a homogenous firm world, because high productive firms should be able to outbid low productive firms. Furthermore, this will make directed search more desirable than in our setting because high productive firms should be able to signal their types in order to get matched with a higher probability. Finally, it would be interesting to consider limited interview capacity. We expect that our main results still hold if firms can interview at most n workers. In the network formation process this makes it less attractive to offer high wages. Since we already find that with full recall, posted wages equal the reservation wage, we expect our results to hold there as well.

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6 Appendix

A Proofs

A.1 Proof of Lemma 1

Consider part (i). We prove the engagement result of (i) by contradiction. Denote the highest number of applicants that a firm in a worker subgraph has by $N^w \leq u$. Note, that all applicants of firms in a worker subgraph are in the same subgraph, since workers in a worker subgraph cannot be linked to a firm in an even or firm subgraph. Suppose firm j is part of a worker subgraph and it is not engaged in $t = N^w$. In $t = N^w$, according to strategy C2a, firm j must have offered the job to all its applicants and must have been rejected by all its applicants. Thus, all workers that are linked to firm j must be engaged with some other firm in the same worker subgraph, since workers always keep their best offer w^h according to strategy A2 and since workers in a worker subgraph are only linked to firms in the same worker subgraph. This leads to the desired contradiction, since the number of engaged workers cannot exceed the number of engaged firms in a worker subgraph.

Now consider the wage result of (i). In round $t = 1$ all firms start with the lowest possible offer, i.e., $w = 0$. According to the engagement result of part (i) all firms in a worker subgraph are engaged in round $t = u < T$. The counter offer strategy B2 rules out that an engaged firm will offer a wage $w \geq \Delta$ if the wage offered by the competing firm is no higher than $w = 0$. Thus, if we can rule out that a rejected firm in a worker subgraph offers $w \geq \Delta$ at any $t < T$, we have proven that the engaged workers of firms in worker subgraphs hold an offer no higher than $w = 0$ in any round $t < T$. The strategy C2b implies that a rejected firm only offers $w \geq \Delta$ in a round $t < T$ if $k = 0$. Since there are more workers than firms in a worker subgraph and since all workers are collectively linked to all firms in the subgraph, there is always at least one applicant without an offer, i.e., $k > 0$. Thus, the wage offers in a worker subgraph implied by the above strategies are no higher than $w = 0$ in any round $t < T$.

Consider now part (ii). We use a contradition argument to rule out that firms in even subgraphs offer a wage $w = 2\Delta$. According to strategy C2b, a rejected firm only offers $w = 2\Delta$ if all its applicants are engaged and hold an offer $w = \Delta$. If all applicants of a rejected firm in an even subgraph are engaged, it must be the case that at least one of the applicants is engaged with a firm outside the even subgraph, because a firm in an even subgraph cannot be rejected, if all workers in even subgraphs are engaged with firms in even subgraphs. The Decomposition Theorem of Corominas-Bosch (2004) implies that workers in an even subgraph are either linked to firms in an even or to firms in a worker subgraphs. Thus, the applicant that is engaged with a firm outside the even subgraph must be engaged with a firm in a worker subgraph. This leads to the desired contradiction, since part (i) of the Lemma implies that the wage offers made by firms in worker subgraphs cannot be higher than $w = 0$. Thus, the rejected firm will according to strategy C2b never offer a wage $w = 2\Delta$ or higher. This implies that wages paid in even subgraphs are no higher than $w = \Delta$. According to strategy A2, since workers keep their best offers, it follows that any firm that offers $w = \Delta$ must be engaged. Since all firms start in round $t = 1$ with the lowest possible offer, i.e., $w = 0$, and since there are at most u workers linked to firms in even subgraphs, it takes at most $t = 2u$ rounds of rejections (where the wage offers $w = 0$ are rejected) until a firm offers for the first time the wage $w = \Delta$ and becomes engaged.

Now consider the engagement result of (iii). Denote the number of workers in a firm subgraph by $u^f \leq u$. Note, that all v^f firms in the respective firm subgraph are only linked to their respective u^f applicants in the same firm subgraph. Thus, since there are more firms than workers in a firm subgraph, i.e., $v^f > u^f$, at least one firm must always be rejected in any round. Strategy C2a implies that the rejected firms first choose one of the k applicants without an offer (if present) and offer her a wage $w = 0$. Thus, it takes at most $t = u^f$ rounds until all workers in a firm subgraph are engaged.

Now consider the wage result of (iii). The engagement result of part (iii) implies that all workers are engaged in round $t = u^f$, i.e., $k = 0$, and at least one firm is rejected. The rejected firm will according to strategy C2b choose one of the applicants with the lowest best offer, i.e., $\underline{w}^h = \min W^N$, and offer this applicant one cent more, i.e., $w = \underline{w}^h + \Delta$ if $\underline{w}^h < 1$. If the offer $w = \underline{w}^h + \Delta$ does not attract a worker, i.e., the rejected firm does not become engaged (which can happen, if the already engaged firm matches the offer according to strategy B2), the firm can offer $w = \underline{w}^h + \Delta$ to other (potential) applicants that hold an offer $\underline{w}^h$. After at most u^f rounds all applicants will hold an offer $\underline{w}^h + \Delta$ and the firm will either be engaged or still remain rejected. Since there is at least one rejected firm each round, wages will increase by Δ after at least u^f rounds, i.e., after all workers experienced wage increases by Δ. By induction firms will increase their offers according to strategy C2b up to $w = 1$. Thus, there exists a round $t \leq u/\Delta$, in which all workers in a firm subgraph hold an offer $w^h = 1$, which they accept. ■

The fact that wages differ across subgraphs enables firms to update their belief on whether they are in a firm subgraph or not. If firms have no applicant, i.e., $N = 0$, they are part of a firm subgraph by definition, i.e., $b_{j,0}(0,.) = 1$. Firms with at least one applicant, i.e., $N > 0$, start with a belief $b_{j,0}\left(N, W^N\right) \in (0,1)$ that is equal to the ex-ante probability to be in a firm subgraph given N. ■

A.2 Proof of Lemma 2

(i) Lemma 1 implies that only firms in firm and even subgraphs observe wage offers $w^h \geq \Delta$ in rounds $t \geq 2u$. Furthermore, at least one firm in each firm subgraph that observes $w^h \geq \Delta$ is rejected. These rejected firms form the belief $b_{j,t}\left(N, \left\{W^{N-m-n},\ \Delta^m, 1^n\right\}\right) = 1$ in any round $t \geq 2u$, because parts (i) and (ii) of Lemma 1 imply that they would not have been rejected if they were in an worker- or even subgraph.

(ii) In round $t = u/\Delta$ all workers in firm subgraphs will have accepted a wage offer $w = 1$. Thus, at $t = u/\Delta$ all firms that are not in firm subgraphs, i.e., observe $W^N = \left\{\text{Ø}^k, 0^{N-k-m}, \Delta^m\right\}$, can infer that they are either in a worker- or even subgraph, i.e., $b_{j,T}\left(N, \left\{\text{Ø}^k, 0^{N-k-m}, \Delta^m\right\}\right) = 0$. ■

From Lemma 1 we know that $T = u/\Delta$ is sufficiently high to ensure that firms can infer whether or not they are in a firm subgraph. We therefore set $T = u/\Delta$.

A.3 Proof of Proposition 1

Consider first the strategies in round T.

Clearly, the workers' strategy A1 to accept the best offer in $t = T$ maximizes the workers' payoff.

Also, the engaged firms' strategy B1 of matching the outside offer in round $t = T$ is profit maximizing, since engaged firms that do not match outside offers would remain idle and earn a profit of zero.

Let us now turn to the strategies C1a and C1b of rejected firms. If there are some applicants without an offer, i.e., $k > 0$, and if the belief that there are other competing firms is equal to zero, i.e., $b_{j,T}\left(N, \left\{\text{Ø}^k, 0^{N-k-m}, \Delta^m\right\}\right) = 0$ at $t = T$, then the action implied by strategy C1a, i.e., the firm should offer a wage $w = 0$ to one of the applicants without an offer, is profit maximizing. This follows from $b_{j,T}\left(N, \left\{\text{Ø}^k, 0^{N-k-m}, \Delta^m\right\}\right) = 0$, i.e., from the fact that the rejected firm believes that all other firms in the same worker or even subgraph are according to Lemma 1 engaged with other workers and will therefore not make an offer to one of the k applicants that does not hold an offer. If the firm observes a set of wage offers that differs from the ones stated in Lemma 1 and has a belief $b_{j,t}\left(N, W^N\right) \in (0,1)$ it is optimal to follow the action implied by the second part of strategy C1a as characterized in Gautier and Moraga-Gonzalez (2004). Finally, if a rejected firm observes that all its applicants hold an offer in round T, the action implied by strategy C1b, i.e., offering $w = 1$ to one of the candidates, is equally profitable as any other action, since the engaged firms will match outside offers (as implied by strategy B1). Thus, the rejected firm cannot do better by deviating from strategy C1b.

Consider now the strategies in any round $t < T$.

The workers' strategy, A2 to keep the best offer w^h is a dominant strategy, because the rejected firms' strategies C2a and C2b imply that keeping a lower offer can lead to a lower payoff for the worker without increasing the chances of receiving better offers in the future.

Consider now strategy B2 for engaged firms. Obviously, a firm can only be engaged, if it offered a wage $w^h \geq 0$ in the past. Due to the tie-breaking rule, which implies that workers prefer their engaged firm over an outside firm in case both firms offer the same wage, matching an outside offer $w^h = 0$ is optimal since it ensures that the firm stays engaged at zero cost. Now consider the different cases if the outside offer satisfies $w^h \geq \Delta$. Denote the highest wage offer that the engaged worker (A) holds by w^h and the highest offer that another applicant holds by $\widetilde{w}^h$. If all other applicants hold the same or a higher offer, i.e., $\widetilde{w}^h \geq w^h$, it is a dominant strategy to match the outside offer of worker A, since it is the least costly way for the firm to stay engaged. It is also optimal to match the outside offer of worker A if one of the other applicants (B) holds an offer $\widetilde{w}^h = w^h - \Delta$, because otherwise the engaged firm must offer applicant B $\widetilde{w}^h + \Delta = w^h$ to have the chance to become engaged. Note, that the firm cannot be sure that it will become engaged (since other firms might also compete for the same worker). If one of the other applicants (B) holds no offer, matching the outside offer of worker A $w^h \geq \Delta$ cannot be optimal. To see this consider the different subgraphs a firm can be in. If the firm is in a worker subgraph, offering the job to applicant B generates profit 1 while matching the outside offer of worker A generates $1 - \Delta$ or less. If the firm is part of an even subgraph, offering the job to applicant B leads to the expected profit $\gamma + (1 - \gamma)(1 - \Delta)$, where $\gamma > 0$ equals the probability that the firm will pay the wage $w = 0$, while matching the outside offer of worker A generates $1 - \Delta$ for sure. If the firm is part of a firm subgraph, profits are driven down to zero and the firm may as well not match the outside offer of applicant A and offer the job to applicant B. Thus, not matching an outside offer $w^h \geq \Delta$, if one of the other applicants holds no offer is weakly dominating. The same is true, if applicant B holds an offer $\widetilde{w}^h < w^h - \Delta$, where $\widetilde{w}^h \geq 0$. To see this, note first that $\widetilde{w}^h \geq 0$ and $\widetilde{w}^h < w^h - \Delta$ imply $w^h > \Delta$. According to strategy C2b a firm (1) offers $w^h > \Delta$ only if $W^N = \left\{ W^{N-k-l-m-n}, \text{Ø}^k, 0^l, \Delta^m, 1^n \right\}$ with $k = l = 0$ and $m > 0$. Lemmas 1 and 2 then imply that firm 1 that offered w^h to worker worker A is part of a firm subgraph. Since a firm in a firm subgraph will eventually pay a wage $w = 1$, it is optimal for the engaged firm not to compete with firm 1 in the firm subgraph, i.e., not to match w^h, but to offer the job to applicant B at the wage $\widetilde{w}^h + \Delta < w^h$.

The strategies C2a and C2b of rejected firms to pick (one of) the applicant(s) with the lowest offer and to offer this applicant the job at the lowest possible wage are also optimal. Any deviation would lead to lower profits. To see this consider deviations depending on the set of wage offers W^N and the type of subgraph the firm is in. Suppose at least one applicant holds no offer, i.e., $W^N = \left\{ W^{N-k-l-m-q}, \text{Ø}^k, 0^l, \Delta^m, 1^q \right\}$ with $k > 0$, and firm (1) chooses in contrast to strategy C2a to offer the job to some engaged worker (A) that holds an offer $w^h \geq 0$. The profit of this deviating strategy will be $1-\Delta$ in case the worker is part of a worker subgraph, since firm 1 has to offer a wage $w = \Delta$ in order to become engaged. However, playing strategy C2a and offering the job to an applicant without an offer ensures according to Lemma 1 a profit of 1. A similar argument implies that a deviation leads to an expected profit of $\gamma + (1-\gamma)(1-\Delta)$ in case firm 1 is part of an even subgraph. If firm 1 is part of a firm subgraph deviating is equally profitable as playing strategy C2a. Thus, without knowing the subgraph, i.e., with belief $b_{j,t}\left(N, W^N\right) \in (0,1)$, action C2a maximizes expected profits. Next, suppose that all applicants hold an offer and at least one applicant holds an offer $w^h = 0$, i.e., $W^N = \left\{W^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q\right\}$ with $k = 0$ and $l > 0$, and the deviating firm chooses in contrast to strategy C2b to offer the job to some engaged worker (A) that holds an offer $w^h \geq \Delta$. Note that Lemma 1 implies that wage offers $w^h = \Delta$ are only observed, if the worker is part of an even or firm subgraph. The profit of this deviation will be $1-2\Delta$ in case the worker is part of an even subgraph, since the deviating firm has to offer a wage $w = 2\Delta$ in order to become engaged. Following strategy C2b and offering the job to an applicant with an offer $w^h = 0$ ensures a profit $1-\Delta$ since wages in an even subgraph are no higher than Δ (see Lemma 1). If the deviating firm is part of a firm subgraph deviating is equally profitable as playing strategy C2b, since profits are equal to zero anyway. Thus, without knowing the subgraph deviating is never profitable. Finally, suppose all applicants hold an offer $w^h \geq \Delta$, i.e., $W^N = \left\{W^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q\right\}$ with $k = l = 0$ and $m \geq 0$, and the deviating firm chooses in contrast to strategy C2b to offer the job to some engaged worker that holds an offer $w^h > \underline{w}^h = \min W^N$. Note that Lemmas 1 and 2 imply that wage offers $w^h \geq \Delta$ are only observed by rejected firms, if they are part of a firm subgraph. Thus, offering the wage $w = \underline{w}^h + \Delta$ (as implied by strategy C2b) or any other wage $w \in \left[\underline{w}^h + \Delta, 1\right]$ generates the same profit, as the wage in a firm subgraph will eventually increase to $w = 1$. To sum up, deviating from strategy C2b without knowing the subgraph yields a strictly lower expected payoff. ■

A.4 Proof of Lemma 3

To be added.

B Simulation algorithm and decomposing a graph into subgraphs

B.1 Simulation algorithm

In our simulations, we apply the following algorithm where step 2 follows Corominas-Bosch (2004) which is based on Hall's marriage theorem.

Decomposition Theorem (Corominas-Bosch, 2004):

(1) Every graph G can be decomposed into a number of firm subgraphs (G_1^f,..., $G_{n_f}^f$), worker subgraphs (G_1^w,..., $G_{n_w}^w$) and even subgraphs (G_1^e,..., $G_{n_e}^e$) in such a way that each

node (firm or worker) belongs to one and only one subgraph and any firm (worker) in a firm-(worker-) subgraph $G_{i,}^{f}(G_{i,}^{w})$ is only linked to workers (firms) in a firm-(worker-)subgraph $G_{j}^{f}(G_{j}^{w})$.

(2) Moreover, a given node (firm or worker) always belongs to the same type of subgraph for any such decomposition. We will write $G = G_{1}^{f} \cup ... \cup G_{n_f}^{f} \cup G_{1}^{w} \cup ... \cup G_{n_w}^{w} \cup G_{1}^{e} \cup ... \cup G_{n_e}^{e}$, with the union being disjoint.

The decomposition algorithm of Corominas-Bosch (2004) works as follows:

Step a: Eliminate all vacancies that did not receive any applicants.

Step b: For $k = 2, ..., v$, identify the groups of k vacancies that are jointly linked to less than k workers. Remove and collect them. We refer to those subgraphs as firm subgraphs.

Step c: Repeat step b but now reverse the role of workers and vacancies. The resulting subgraphs are called worker subgraphs.

Step d: When all worker subgraphs are removed, the remaining ones are balanced (or even) subgraphs (with an equal number of workers and firms).

Denote the total number of firm subgraphs by F, the total number of worker subgraphs by W, and the number of even subgraphs by E, u_i^f (u_i^e) is number of workers in firm (even) subgraph i, v_i^w is number of firms in worker subgraph i. v_i^f is the number of firms in firm subgraph i, u_i^w is number of workers in worker subgraph i. The number of matches, M, is then given by,

$$M = \sum\nolimits_{i=1}^{F} u_i^f + \sum\nolimits_{i=1}^{W} v_i^w + \sum\nolimits_{i=1}^{E} u_i^e,$$

the fraction of firms in firm subgraphs and the fraction of workers in worker subgraphs by,

$$\frac{v^f}{v} = \sum\nolimits_{i=1}^{F} \frac{v_i^f}{v} \quad \text{and} \quad \frac{u^w}{u} = \sum\nolimits_{i=1}^{W} \frac{u_i^w}{u}.$$

B.2 Simulation examples

To illustrate that having equal application arrival rates is desirable when agents do not know the network we numerically compare equal application rates with the case where a subset of vacancies has a higher application arrival rate. Take a, u, v as given and let $a < v$. First, we color a fraction q of the vacancies blue and a fraction $(1-q)$ green and let each worker send one application to a blue vacancy and the other $(a-1)$ applications to a green one. Each blue vacancy receives an application from worker 1 with probability $1/qv$ and the same for workers 2,..., u. For the $a = 3$ example, each green vacancy gets with probability, $(a-1)/(1-q)v$, the second application of worker 1 and if it did not get the second one, it gets the third one with probability $(a-2)/((1-q)v-1)$ etc. The same holds for the other workers. For $q = 1/a$, the arrival rate at each firm is the same and the only difference with full equalization of application rates is that the market is partitioned. Since we want to focus on network formation here, we assume maximum matching on each realized network and use the Decomposition algorithm given above. Let p_n be the probability that a firm receives no workers, let var(M) be the variance of applicants that a particular firm receives. Recall that the fraction of firms in firm subgraphs is v^f/v and the fraction of workers in worker subgraphs is u^w/u. The matlab code is available upon request.

In Table 1 we present simulation results for $v = u = 12$. We generate a sample of 1000 networks for each case. Table 1 presents those variables for different values of a, q. We see that partitioning the market reduces the expected number of matches, $E(M)$, but that for $q = \frac{1}{a}$ (those rows are in bold), the arrival rate at each firm is the same and the difference in the expected number of matches $E(M)$ with the non partitioning case is relatively small. We also see that if a is large relatively to v, that partitioning hardly matters.

a	p_n	$E(M)$	var(M)	v^f/v	u^w/u
joint					
2	1.343	10.416	0.812	0.012	0.061
3	0.377	11.554	0.382	0.045	0.343
6	0.003	11.997	0.003	0.000	0.003
partitioned ($q = \frac{1}{3}$)					
2	1.748	10.064	0.875	0.187	0.684
3	**0.405**	**11.533**	**0.387**	**0.046**	**0.347**
6	0.124	11.876	0.111	0.010	0.122
partitioned ($q = \frac{1}{6}$)					
2	2.851	9.137	0.945	0.242	0.675
3	0.719	11.206	0.540	0.075	0.510
6	**0.005**	**11.995**	**0.005**	**0.000**	**0.005**

Table 1: Simulation results for $v = u = 12$