Probability Weighting as Evolutionary Second-best*

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Abstract

The economic concept of the second-best involves the idea that multiple simultaneous deviations from a hypothetical first-best optimum may be optimal once the first-best itself can no longer be achieved, since one distortion may partially compensate for another. Within an evolutionary framework, we apply this concept to behavior under risk. We argue that the two main components of prospect theory, the value function and the probability weighting function, are complements in the second-best sense. Previous work has shown that an adaptive S-shaped value function may be evolutionarily optimal if decision-making is subject to cognitive or perceptive constraints. We show that distortions in the way probabilities are perceived can be seen as complementary to this value function. The second-best optimum involves overweighting of small and underweighting of large probabilities. Behavior as described by prospect theory might therefore be evolution's second-best solution to the fitness maximization problem. We also discuss under which circumstance our model makes empirically testable predictions about the relation between individuals' value and probability weighting functions.

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1 Introduction

Lipsey and Lancaster (1956) introduced the concept of the second-best in economics, which is now widespread. It compares the *first-best* solution to an optimization problem under constraints that are "in the nature of things" (p. 12), such as technological constraints for a problem of welfare maximization, to the *second-best* solution when additional constraints are imposed on the same problem. The scope of such additional constraints ranges from exogenously fixed taxes or tariffs (Lipsey and Lancaster 1956) to asymmetric information in contract-theoretic applications (Bolton and Dewatripont 2005). Lipsey and Lancaster's most important finding is "that if one of the Paretian optimum conditions cannot be fulfilled a second best optimum situation is achieved only by departing from all other optimum conditions." (p. 12). For instance, once several countries have introduced tariffs, a unilateral tariff reduction by a single country may well decrease welfare, even though the hypothetical first-best involves no tariffs at all (Viner 1999). Waldman (1994) has first proposed to apply similar arguments in an evolutionary context. In this paper, we aim at translating the second-best concept to evolutionarily optimal behavior under risk.

As a starting point we observe that evolution can be interpreted as a process that maximizes a suitably defined notion of fitness (Maynard Smith 1978), by selecting those behavioral traits that are most conducive to reproduction. Much recent work has employed this paradigm fruitfully in the study of human behavior and its foundation (see Robson and Samuelson (2010) for an overview). With respect to behavior under risk, for instance, Robson (1996) has shown that the largest fitness can be achieved by expected utility maximizing agents, whenever risk is idiosyncratic.[1] This result has been derived in a framework that could be described as first-best, where no constraint besides the natural scarcity of resources has been imposed. Several other contributions (Friedman 1989, Robson 2001, Rayo and Becker 2007, Netzer 2009, Kornienko 2011) have explicitly modelled cognitive or perceptive constraints of a decision-maker. Within different frameworks they conclude that the use of an adaptive, possibly S-shaped value function as postulated in prospect theory (Kahneman and Tversky 1979) is superior in the presence of such constraints.[2] In short, steepness of the function that evaluates payoffs is helpful to prevent decision mistakes in these models, because it enables to distinguish even alternatives that are very similar to each other. Hence a relatively large slope should be allocated to regions where correct decisions matter most

[1] This is no longer the case with correlated risks. See Cooper and Kaplan (1982), Robson (1996), Bergstrom (1997), Curry (2001) and Robson and Samuelson (2009).

[2] See also Hamo and Heifetz (2002) for a model in which S-shaped utility evolves in a framework with aggregate risk, and McDermott et al. (2008) for a model where S-shaped utility is optimal if reproductive success is determined by a single fitness threshold.

(the agent's reference point), which explains the adaptive nature and the S-shape of the value function.

The above papers all assume that the decision-maker perceives probabilities correctly. This assumption seems not justified. To the contrary, even if there is no constraint that directly prevents the correct perception of probabilities, the general concept of the second-best outlined above leads us to the conjecture that probabilities should *not* be perceived correctly. We investigate the validity of this conjecture in a simple model where nature offers randomly drawn prospects (lotteries) to an agent, who decides whether to accept or reject. For instance, the choice whether or not to go hunting in a specific area, at a particular time, is one such decision that humans were regularly confronted with during evolutionary history. The eventual outcome of a hunt is stochastic, with varying, situation-dependent probabilities. We now take the insights of the above discussed literature for granted and assume that, to evaluate payoffs, the agent uses an adaptive and S-shaped, rather than a fixed fitness-reflecting value function. Hence we do not model underlying cognitive or perceptive constraints explicitly. This shortcut allows us to highlight the relation between value and probability weighting function most transparently. Importantly, we also do not assume that constraints directly prevent the correct perception of probabilities; correct perception would in principle be possible, so that any probability weighting that emerges in our model is solely an optimal adaptation to the value function.

The basic model and our main results will be presented in Section 2, where we consider simple prospects (with only one possible gain and one possible loss) and assume away loss aversion. As a benchmark we first allow perceived probabilities to depend not only on true probabilities but also on the associated payoffs, and we show that first-best behavior, i.e. expected fitness/utility maximizing behavior can then be implemented despite the distortion introduced by the value function. We then impose the restriction that probability weighting has to be payoff-independent, in line with the usual descriptive theories for decisions under risk. In that case, the first-best can no longer be achieved generically. Furthermore, any solution to the fitness maximization problem involves *overweighting* of small and *underweighting* of large probabilities. Intuitively, lotteries with an expected fitness close to zero are especially prone to decision mistakes, and small probabilities go along with large absolute payoffs in such lotteries. Large absolute payoffs are, in turn, relatively undervalued by an S-shaped value function. To compensate, it becomes optimal to overweigh small and underweigh large probabilities. We show that the optimal probability perception can be implemented by reflective and symmetric weighting (Prelec 1998) in some cases, employing weighting functions that have been used in both empirical and theoretical work on prospect theory (Gonzalez and Wu 1999).

Section 3 contains several extensions of the model. First, we show that the introduction of loss aversion – the systematically different treatment of gains and losses – implies that gain and loss probabilities should also be perceived systematically different. We predict violations of reflectivity in the direction of an optimism bias (Sharot et al. 2007). Second, we illustrate that our results do not depend on the use of simple prospects. For more complicated prospects, our model predicts behavior in line with rank-dependent models (Quiggin 1982). Finally, in Section 4 we briefly discuss the issue of empirical testability of our results, and Section 5 contains concluding remarks.

In Friedman and Massaro (1998) a reason for overestimation of small and underestimation of large probabilities is suggested which is very different from our argument. Agents cannot perceive (or process) the true objective probability $p \in [0, 1]$ but instead have to work with $\hat{p} \in [0, 1]$ which corresponds to the true probability p plus a noise term e. The signalling process is such that if $|p + e| > 1$ then a new signal is drawn, and thus the expected value of the true probability $E(p|\hat{p})$ is closer to 0.5 than the signal $\hat{p}$. This way of dealing with probabilities is completely rational once the signal generating process is taken into account.

There are several other contributions that can be classified as belonging to the evolutionary second-best approach to behavior. They all share the property that some underlying constraint induces one distortion, or behavioral anomaly, which in turn makes it optimal to introduce another anomaly to partially compensate for the first. In Samuelson (2004) and Noeldeke and Samuelson (2005), agents cannot correctly process information about their environment, specifically about current earnings opportunities. Concern for relative consumption then becomes a second-best optimal way of utilizing the information inherent in others' consumption levels. Samuelson and Swinkels (2006) argue that choice-set dependency might be an analogous way of correcting for an agent's lack of understanding of her choices' fitness implications under different conditions. Baliga and Ely (2011), although not explicitly within an evolutionary framework, start from the assumption of imperfect memory: after having made an investment, an agent forgets about the details of the project. The initial, sunk investment now still contains information, and relying on this information for future decisions on the project can be second-best, rather than a sunk cost "fallacy". Finally, Suzuki (2011) investigates the joint optimality of present bias, false beliefs and a concern to avoid cognitive dissonance.

Before we present our model in detail, we want to address an obvious question: Why would evolution introduce additional anomalies rather than remove the origin of the problem?[3] First, it might not be able to do so because the underlying constraint cannot be relaxed. For instance, a mutation with weaker cognitive constraints might exhaust too much

[3]See also Ely (2011) for a discussion of this question.

energy to still be competitive. Second, even if the original problem could be removed, it is possible that an evolutionary process gets stuck in a second-best. In Waldman (1994), a first-best mutant will not be able to invade a second-best population because its optimal characteristics are diluted through the process of sexual recombination.[4] This argument does not require the existence of cognitive constraints at all. A different argument is given by Ely (2011), who considers a complex species that reproduces asexually and whose behavior is determined by many different parameters. Even though all mutations are possible, i.e. the probability of a mutation to the first-best parameter setting is strictly positive, the species can remain second-best forever, provided that smaller improvements at only a few parameters ("kludges") are more likely and the species' complexity grows sufficiently fast, which makes an immediate mutation to the first-best increasingly unlikely.

2 Probability Weighting for Simple Prospects

2.1 The Basic Model

Consider the following model. There is an agent who has a current level of fitness $c \in \mathbb{R}$ (which could change over time), that serves as her reference point. That is, she evaluates payoffs as gains and losses relative to c. In doing so, we assume that the agent makes use of an S-shaped value function, which is denoted $V : \mathbb{R} \to \mathbb{R}$ and depicted in the following figure.

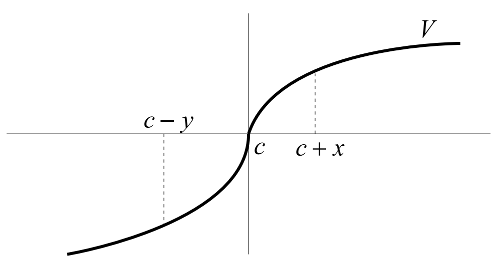

Figure 1: The Value Function

Adaptation to the reference point is captured by the assumption that $V(c) = 0$. It is convenient to write both gains and losses as positive numbers, and to decompose V into two functions, one used to evaluate gains (v_G) and one to evaluate losses (v_L). Specifically, we define $v_G : \mathbb{R}_+ \to \mathbb{R}_+$ by $v_G(x) = V(c + x)$ for all gains $x > 0$, and $v_L : \mathbb{R}_+ \to \mathbb{R}_+$ by $v_L(y) = -V(c - y)$ for all losses $y > 0$. We assume that both v_G and v_L are differentiable,

[4]See Dobbs and Molho (1999) for a generalized version of the model by Waldman (1994).

strictly increasing and strictly concave, and we keep them fixed throughout this section. As outlined in the Introduction, earlier work has tried to derive the value function from more basic principles, such as cognitive or perceptive constraints. Because the focus of this paper is to illuminate the interplay between probability weighting and the value function, we refrain from providing a foundation of the value function itself, but simply take it as given. We would like to point out, however, that arguments such as those given by Waldman (1994) or Ely (2011) provide an additional justification for our approach, even when the value function remains unsubstantiated. We will return to this issue in Section 4.

We start by considering *simple prospects*. Such prospects consist of one possible gain of size $x > 0$ which occurs with probability $0 < p < 1$, and one possible loss of size $y > 0$ which occurs with the opposite probability $1 - p$. It turns out to be convenient to define such prospects in terms of relative rather than absolute probabilities. Hence a simple prospect is a tuple (q, x, y) from the set $\mathscr{P} = \mathbb{R}^3_+$, where $q = p/(1-p)$ is the relative probability of a gain. We assume that gains and losses are measured in terms of biological fitness, which facilitates the analysis.[5] We can then define a fitness function $F : \mathscr{P} \to \mathbb{R}$ that assigns to each prospect $(q, x, y) \in \mathscr{P}$ its expected fitness

$$F(q,x,y) = \left(\frac{q}{1+q}\right) x - \left(\frac{1}{1+q}\right) y \tag{1}$$

which, relying on the results by Robson (1996), we take as the relevant criterion for evolutionary success.

The basic choice situation is as simple as possible: when the agent is faced with a prospect, she has to decide whether to accept or reject it. Then, (q, x, y) should be accepted if $F(q, x, y) \geq 0$, or equivalently $q \geq y/x$, so that it (weakly) increases expected fitness above the current level. In a dynamic interpretation, $c+x$ or $c-y$ could become the new reference point if the prospect has been accepted and the risk has realized. Let $\mathscr{P}^+ = \{(q, x, y) \in \mathscr{P} | q \geq y/x\}$ be the set of prospects with weakly positive and $\mathscr{P}^- = \mathscr{P} \backslash \mathscr{P}^+$ of those with negative expected fitness.

To start with the most general approach, let $\eta : \mathscr{P} \to \mathbb{R}_+$ be a weighting function that, for any prospect $(q, x, y) \in \mathscr{P}$, yields a subjective *relative decision weight* $\eta(q, x, y)$. Thus, given a prospect (q, x, y), the agent perceives the relative gain probability q as $\eta(q, x, y)$. Equivalently, the actual gain probability $p = q/(1+q)$ is perceived as $\eta(q, x, y)/(1+\eta(q, x, y))$, and the loss probability $1 - p$ as $1/(1 + \eta(q, x, y))$ accordingly. We will be interested in the weighting function that maximizes expected fitness given the fixed value function, with the

[5] Defining payoffs in terms of, say, consumption levels adds an additional layer and makes it necessary to elaborate on the way consumption translates into fitness.

second-best-idea in mind that evolution should have selected for agents whose anomalies optimally compensate one another. Denote the set of weighting functions evolution can choose from by $\mathscr{W} \subseteq \mathbb{R}_+^{\mathscr{P}}$. Then, we can define a functional $U : \mathscr{P} \times \mathscr{W} \to \mathbb{R}$ that assigns the subjective utility score

$$U_\eta(q,x,y) = \left(\frac{\eta(q,x,y)}{1+\eta(q,x,y)}\right) v_G(x) - \left(\frac{1}{1+\eta(q,x,y)}\right) v_L(y) \tag{2}$$

to prospects, using perceived probabilities as given by η. The agent will accept prospect (q,x,y) if and only if $U_\eta(q,x,y) \geq 0$. This can be reformulated to $\eta(q,x,y) \geq v_L(y)/v_G(x)$, which is directly comparable to the criterion for optimal choice $q \geq y/x$. Given that the value function distorts one side of the optimality condition, it becomes apparent that the other side should be distorted as well.

We conclude the introduction of the model by specifying the way in which prospects are offered to the agent by nature. We assume that nature randomly draws and offers to the agent one prospect at a time, according to a probability distribution that can be described by a strictly positive density h on $\mathscr{P}$.[6] The agent, endowed with a fixed weighting function η, then evaluates the offered prospect according to U_η and decides whether to accept or to reject. Let $\mathscr{P}_\eta^+ = \{(q,x,y) \in \mathscr{P} | \eta(q,x,y) \geq v_L(y)/v_G(x)\}$ be the set of prospects that the agent accepts, and $\mathscr{P}_\eta^- = \mathscr{P} \backslash \mathscr{P}_\eta^+$ the set of rejected prospects. The optimal, expected fitness maximizing weighting function is then determined by the following program:

$$\max_{\eta \in \mathscr{W}} \int\limits_{\mathscr{P}_\eta^+} F(q,x,y) h(q,x,y) d(q,x,y). \tag{3}$$

Problem (3) has the following equivalent formulation (see Appendix A.1 for details):

$$\min_{\eta \in \mathscr{W}} \int\limits_{(\mathscr{P}_\eta^- \cap \mathscr{P}^+) \cup (\mathscr{P}_\eta^+ \cap \mathscr{P}^-)} |F(q,x,y)| h(q,x,y) d(q,x,y), \tag{4}$$

which prescribes the minimization of (non-negative) fitness lost due to rejection of positive fitness prospects (from the set $\mathscr{P}_\eta^- \cap \mathscr{P}^+$) and acceptance of negative fitness prospects $(\mathscr{P}_\eta^+ \cap \mathscr{P}^-)$. Formulation (4) is convenient for the following reason. Suppose there exists a weighting function $\eta \in \mathscr{W}$ for which the objective in (4) takes the value zero. Given that h is strictly positive, this requires that $\mathscr{P}_\eta^+ = \mathscr{P}^+$, so that $\mathscr{P}_\eta^- \cap \mathscr{P}^+ = \mathscr{P}_\eta^+ \cap \mathscr{P}^- = \varnothing$ (except for measure-zero sets of prospects). Hence behavior under η is identical to expected

[6] The assumption of a continuous distribution is not essential to our arguments. The full support assumption makes sure that there is actually evolutionary pressure for optimal handling of all possible prospects.

fitness/utility maximization, i.e. η perfectly compensates for the value function. We say that the first-best is achievable in this case. Otherwise, if the optimal value of (4) is strictly positive, the solution is in fact only second-best, with observed behavior that systematically deviates from unconstrained expected fitness maximization.

2.2 Payoff-dependent Weighting

As a benchmark, we first want to consider the possibility of unconstrained choice of η. Let $\mathscr{F} = \mathbb{R}_+^{\mathscr{P}}$ be the complete set of weighting functions, which includes functions where perceived probabilities do not only depend on true probabilities but also on the size of the associated gain and loss.[7] We can then state the following immediate result.

Proposition 1. *Suppose $\mathscr{W} = \mathscr{F}$. Then, the first-best is achievable.*

Consider the weighting function

$$\eta^{FB}(q,x,y) = q \cdot \frac{v_L(y)}{v_G(x)} \frac{x}{y},$$

for instance, for which we immediately obtain $\eta^{FB}(q,x,y) \geq v_L(y)/v_G(x) \Leftrightarrow q \geq y/x$, so that actual and fitness maximizing decisions coincide. We will not discuss this function in further detail, as the behavior it induces cannot be distinguished from expected utility maximization.[8] One aspect, however, is worth being pointed out: η^{FB} is linear in q for given x and y, so that there is no need to systematically distort the relative perception of large and small probabilities when weighting can condition on payoffs.

2.3 Separate Weighting

In prospect theory applied to simple prospects (Kahneman and Tversky 1979), probability weighting is assumed to be independent of payoffs, i.e. perceived probabilities depend on true probabilities but not on the size of gains and losses. The same holds true for expected utility theory (von Neumann and Morgenstern 1944) where $\eta(q,x,y) = q$, in terms of our notation. In how far the human brain processes probabilities separately from the evaluation of gains and losses remains an investigated question in neuroscience.[9] Some support for this

[7] Fehr-Duda et al. (2010) report an experimental instance of non-separability of payoffs and probabilities.

[8] See also Acemoglu and Yildiz (2001) for an evolutionary model of strategic interactions in which different behavioral anomalies emerge that perfectly compensate for one another.

[9] There are several further neuroscience studies that have also examined the neural basis of probability weighting. Tobler et al. (2008), for instance, investigate the coding of probability perception for non-choice situations. They also provide a detailed literature review. See also Zhong et al. (2009a) and (2009b).

assumption is provided by Berns et al. (2008). In an fMRI study they find behaviorally meaningful neural correlates of non-linear (inverse S-shape) probability weighting. They show that "the pattern of activation could be largely dissociated into magnitude-sensitive and probability-sensitive regions" (p. 2052), which they explicitly interpret as evidence for the hypothesis that "people process these two dimensions separately" (p. 2055).[10] Hence let

$$\mathscr{F}' = \{\eta \in \mathscr{F} | \exists\, \tilde{\eta} \in \mathbb{R}_+^{\mathbb{R}_+} | \forall (q,x,y) \in \mathscr{P}, \eta(q,x,y) = \tilde{\eta}(q)\}$$

be the set of relative weighting functions which are independent of payoffs, and write, with slight abuse of notation, such functions as functions $\eta(q)$ of q only. We will continue to derive optimality conditions for η if choice is restricted to $\mathscr{F}'$. The relation of our findings to prospect theory will be discussed in the next subsection.

First, we focus on the symmetric case without loss aversion. Formally, assume that the functions v_G and v_L are identical, so that gains and losses are evaluated symmetrically. We denote this common valuation function by v. The following proposition states the important property of optimal weighting that the relative probability of the gain is undervalued if and only if the gain is the more likely than the loss.

Proposition 2. *Suppose $\mathscr{W} = \mathscr{F}'$ and there is no loss aversion. Then, any solution η^* to (4) satisfies*

$$\eta^*(q) \gtreqqless q \quad \Leftrightarrow \quad q \lesseqqgtr 1, \tag{5}$$

for almost all $q \in \mathbb{R}_+$.

Proof. See Appendix A.2. □

To grasp an intuition for the result, suppose the agent perceives probabilities correctly ($\eta(q) = q$). Assume nature offers a prospect (q, x, y) where $y = qx$, so that the prospect's fitness is exactly zero. Will the agent accept or reject it? The condition for rejection becomes $qv(x) < v(qx)$ which, due to strict concavity of v, is equivalent to $q < 1$. Hence whenever a zero fitness prospect has a small relative gain probability, it will be rejected when probabilities are perceived correctly. The reason is that any such prospect must satisfy $y < x$, i.e. its gain must be larger than its loss. But then the use of the concave function v implies that the gain is undervalued relative to the loss, resulting in rejection. Monotonicity of v implies that

[10]The only region that Berns et al. (2008) found activated by both payoffs and probabilities is an area close to the anterior cingulate cortex, which they see as a "prime candidate for the integration of magnitude and probability information" (p. 2055). Note however that there exists also neuroscience evidence of neural activity responding both to changes in probabilities and changes in gains, e.g. Platt and Glimcher (1999) studying monkey neurons in the lateral intraparietal area, an area of primate cortex that is thought to transform visual signals into eye-movement commands.

all negative fitness lotteries with $q < 1$ are also rejected, but the same still holds for some positive fitness lotteries, by continuity of v. Put differently, when there is no probability weighting, prospects (q, x, y) with $q < 1$ are subject to only one of the two possible types of mistakes: rejection of prospects with strictly positive fitness. The analogous argument applies to prospects with $q > 1$, where correct probability perception implies that only the mistake of accepting a negative fitness prospect can occur. To counteract these mistakes, it becomes optimal to overvalue small relative gain probabilities ($\eta^*(q) > q$ if $q < 1$) and to undervalue large relative gain probabilities ($\eta^*(q) < q$ if $q > 1$).

Note that neither Proposition 2 nor the above intuition depend on the prior h. The exact shape of the solution η^* will generally depend on h, but the direction of probability weighting given in (5) does not. Hence Proposition 2 does not rely on (unknown) details about the environment in which evolution operated.

Can the first-best still be achieved if choice of η is restricted to $\mathscr{F}'$? And if so, what is the first-best weighting function? We are able to answer these two questions as follows.

Proposition 3. *Suppose $\mathscr{W} = \mathscr{F}'$ and there is no loss aversion. Then, the first-best is achievable if and only if $v(z) = \beta z^\alpha$ for some $\beta > 0$ and $0 < \alpha < 1$. The corresponding first-best weighting function is $\eta^{FB}(q) = q^\alpha$.*

Proposition 3 is a special case of Proposition 4, which we present in Section 3.1 and prove in Appendix A.5. Achieving the first-best requires to align actual and fitness-maximizing behavior. Specifically, all fitness-neutral prospects (q, x, qx) must actually be identified as such by the agent, so that, by monotonicity, all prospects with strictly positive fitness will be accepted and those with strictly negative fitness will be rejected. The condition $U_\eta(q, x, qx) = 0$ can be reformulated to $\eta(q) = v(qx)/v(x)$. This can only be solved by some $\eta \in \mathscr{F}'$ if the ratio $v(qx)/v(x)$ is independent of x, for all q. It can be shown that any continuous, strictly increasing and concave function v which satisfies this property must be of the CRRA form $v(z) = \beta z^\alpha$. Whenever the value function does not belong to this class (which should be considered as the generic case), the fitness-maximizing solution η^* is truly second-best. As a consequence, the agent's behavior will deviate from expected fitness maximization, in the direction predicted by prospect theory, as the next section will show.

2.4 Relation to Prospect Theory

We now want to examine how the above derived optimality conditions relate to the probability weighting postulated in prospect theory. There, it is usually assumed (Prelec 1998) that a function $\pi_G : [0, 1] \to [0, 1]$ transforms the actual gain probability p into a perceived gain decision weight $\pi_G(p)$, and a function $\pi_L : [0,1] \to [0,1]$ likewise transforms the loss probability $1-p$ into $\pi_L(1-p)$. The term *decision weights* is used because $\pi_G(p)$ and $\pi_L(1-p)$ do not necessarily sum up to one, and thus are not necessarily probabilities.

The *absolute* weighting functions π_G and π_L relate to our *relative* weighting function η as follows. On the one hand, any given pair (π_G, π_L) induces a unique η, i.e. when π_G and π_L are employed, the relative gain probability $q = p/(1-p)$ is perceived as

$$\eta_{(\pi_G,\pi_L)}(q) = \frac{\pi_G(q/(1+q))}{\pi_L(1/(1+q))}.$$

The converse is not true, because there are different pairs of absolute weighting functions (π_G, π_L) that implement a given relative perception η. That means there is not a unique way of representing the optimal weighting η^* in terms of prospect theory weighting functions, except if we impose additional requirements on (π_G, π_L). Two candidates that have received attention in both the theoretical and the empirical literature are the following (Prelec 1998):

1. *Reflectivity.* (π_G, π_L) is reflective if $\pi_G(r) = \pi_L(r)$, $\forall r \in [0,1]$.

2. *Symmetry.* (π_G, π_L) is symmetric if $\pi_i(1-r) = 1 - \pi_i(r)$, $\forall r \in [0,1]$ and $i = G, L$.

Reflectivity requires the identical treatment of gain and loss probabilities. We will skip the indices G and L when referring to reflective weighting functions. Reflective weighting is especially appealing from an evolutionary perspective. It redundantizes the maintenance of two separate weighting mechanisms, which would be resource expensive and could lead to undesirable interference effects. Hence evolution should have settled on reflective weighting if the optimal probability perception can be implemented in this way. Even if no reflective probability weighting function implements our second-best solution, evolution may favor a reflective probability weighting function if the savings in cognitive resources outweigh the expected loss from suboptimal decisions. Symmetry is then also appealing because, if imposed in addition to reflectivity, it ensures that $\pi(p) + \pi(1-p) = \pi(p) + 1 - \pi(p) = 1$ for all $p \in [0,1]$, i.e. that probabilities are in fact perceived as probabilities. Let us see whether the optimal η^* can be implemented by reflective weighting, and how property (5) then affects the shape of the absolute weighting function.

Lemma 1. *(i) A function $\eta \in \mathscr{F}'$ can be implemented by reflective weighting if and only if*

$$\eta(1/q) = 1/\eta(q), \forall q \in \mathbb{R}_+. \tag{6}$$

(ii) Suppose the optimal weighting function η^* derived from program (4) satisfies (6). Then

any reflective π_{η^*} that implements η^* satisfies

$$\frac{\pi_{\eta^*}(p)}{\pi_{\eta^*}(1-p)} \gtreqless \frac{p}{1-p} \Leftrightarrow p \lesseqgtr \frac{1}{2}. \tag{7}$$

If symmetry is imposed in addition, $\pi_{\eta^*} = \eta^*(p/(1-p))/(1+\eta^*(p/(1-p)))$ is unique, with

$$\pi_{\eta^*}(p) \gtreqless p \Leftrightarrow p \lesseqgtr \frac{1}{2}. \tag{8}$$

Proof. See Appendix A.3. □

Prelec (1998) summarizes empirical evidence on the shape of weighting functions, which provides support for reflectivity. Our Lemma 1 identifies condition (6) as crucial for the question whether or not the evolutionarily optimal weighting can be implemented reflectively. For instance, $\pi^{FB}(q) = q^\alpha$ presented in Proposition 3 satisfies (6). According to the lemma, the unique reflective and symmetric weighting function that implements η^{FB} is given by

$$\pi_{\eta^{FB}}(p) = \frac{p^\alpha}{p^\alpha + (1-p)^\alpha}. \tag{9}$$

Function (9) has been studied in the context of probability weighting by Karmarkar (1978, 1979). In Figure 2, it is depicted by a solid line for the case when $\alpha = 1/2$.

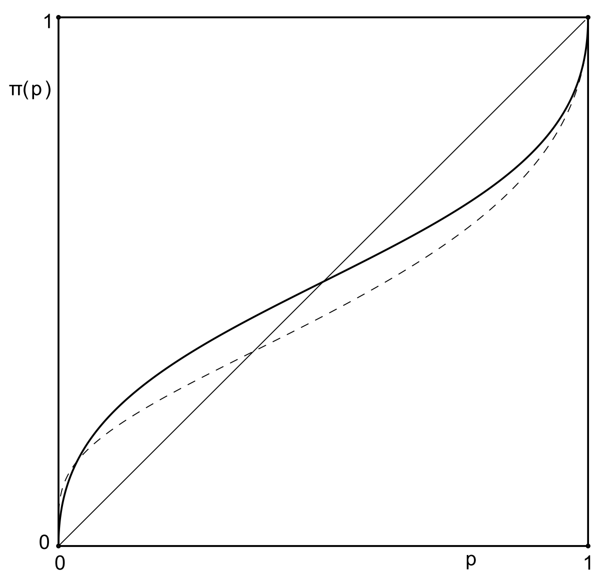

Figure 2: Common Weighting Functions.

In principle, (8) corresponds to a key property of probability weighting functions in prospect theory: small probabilities are over-weighted and large probabilities are under-weighted. However, there is much evidence on the *asymmetry* of weighting functions. Individuals tend to overweigh absolute probabilities below approximately 1/3 and underweigh those above 1/3, in contrast to (8). Hence symmetry might not be a reasonable property to impose. Prelec (1998) instead suggests and axiomatizes the following class of reflective but non-symmetric functions:

$$\pi(p) = e^{-[(-\ln p)^{\alpha}]} \quad \text{where} \quad 0 < \alpha < 1. \tag{10}$$

We show in Appendix A.4 that this function indeed satisfies condition (7). This does, of course, not yet imply that (10) is optimal. It only shows that the function suggested by Prelec remains a candidate for optimal weighting, as it satisfies a necessary optimality condition. In fact, the relative weighting implemented by function (10) is given by

$$\eta(q) = \frac{e^{[\ln(1+q)]^{\alpha}}}{e^{[\ln(1+q)-\ln(q)]^{\alpha}}},$$

which is similar but not identical to $\eta^{FB}(q) = q^{\alpha}$, so that (10) remains a candidate for actual second-best weighting only. The dashed line in Figure 2 depicts (10), for the exponent 1/2.

3 Extensions

3.1 Loss Aversion

As a first extension, we want to allow for loss aversion, which means that the agent takes a loss of given size more seriously than a gain of the same size. We formalize this as follows. We assume that there exists a function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that $v_G(z) = v(z)$ and $v_L(z) = \gamma v(z)$ for some $\gamma \geq 1$. The parameter γ measures the degree of loss aversion. For $\gamma = 1$ we obtain the special case already considered in Proposition 3, while loss aversion requires $\gamma > 1$. We can then state the following generalization of Proposition 3.

Proposition 4. *Suppose $\mathscr{W} = \mathscr{F}'$ and $\gamma \geq 1$. Then, the first-best is achievable if and only if $v(z) = \beta z^{\alpha}$ for some $\beta > 0$ and $0 < \alpha < 1$. The corresponding first-best weighting function is $\eta^{FB}(q) = \gamma q^{\alpha}$.*

Proof. See Appendix A.5. □

There are two main insights that can be derived from Proposition 4. First, the fact that the first-best can be achieved for special cases but not in general is unaffected by the existence of loss aversion. The condition for attainability of the first-best is independent of the size of γ. For the second main insight, consider the optimal $\eta^{FB}(q) = \gamma q^\alpha$ and its implementation in terms of absolute weighting functions (π_G, π_L). Whenever $\gamma > 1$, we have $\eta^{FB}(1/q) = \gamma/q^\alpha \neq 1/\gamma q^\alpha = 1/\eta^{FB}(q)$, so that (6) is violated and the optimum can no longer be implemented reflectively. In fact, both reflectivity *and* symmetry must be violated. This is true in much greater generality, i.e. not only when the first-best is attainable, as the following proposition shows.

Proposition 5. *Suppose* $\mathscr{W} = \mathscr{F}'$ *and* $\gamma > 1$. *Let* η^* *be a solution to (4), and assume* (π_G^*, π_L^*) *implements* η^*. *Then,* (π_G^*, π_L^*) *is neither reflective nor symmetric.*

Proof. Let $\mathscr{W} = \mathscr{F}'$, $\gamma > 1$, and suppose η^* is a solution to (4). Arguing as in the proof of Proposition 2, it then follows that $\eta^*(q) > q$ for all $q < \gamma$. In particular, since $\gamma > 1$, we have $\eta^*(1) > 1$. Assume (π_G^*, π_L^*) implements η^*, i.e. $\pi_G^*(q/(1+q))/\pi_L^*(1/(1+q)) = \eta^*(q)$. For $q = 1$ we obtain $\pi_G^*(1/2)/\pi_L^*(1/2) = \eta^*(1) > 1$, and thus $\pi_G^*(1/2) > \pi_L^*(1/2)$. Hence (π_G^*, π_L^*) is not reflective. Furthermore, symmetry requires $\pi_i(1/2) = 1/2$ for both $i = G, L$, so that (π_G^*, π_L^*) is also not symmetric. □

On a positive account, the model with loss aversion does actually provide a basis for asymmetric weighting, in line with empirical findings. Yet, loss aversion also implies non-reflectivity in our model, which seems not supported by the empirical evidence. If there is an asymmetry in the evaluation of gains and losses, the second-best principle calls for a systematically different treatment of gain and loss probabilities. Let us examine the properties of this differential treatment more closely. Consider the solution $\eta^{FB}(q) = \gamma q^\alpha$, for instance. The following absolute weighting functions, called "linear in log odds" by Gonzalez and Wu (1999), implement η^{FB} and additionally satisfy $\pi_G^{FB}(p) + \pi_L^{FB}(1-p) = 1$:

$$\pi_G^{FB}(p) = \frac{\gamma p^\alpha}{\gamma p^\alpha + (1-p)^\alpha}, \qquad \pi_L^{FB}(p) = \frac{(1/\gamma)p^\alpha}{(1/\gamma)p^\alpha + (1-p)^\alpha}. \tag{11}$$

They are illustrated in Figure 3 for the case of $\alpha = 1/2$ and $\gamma = 3/2$.[11] Importantly, the function used to weight gain probabilities lies above the one for losses, which compensates for the relative overassessment of losses implied by loss aversion. Hence the model predicts a specific type of non-reflectivity, which is close to what has become known as overconfidence (Camerer and Lovallo 1999) or, even more closely, optimism (Sharot et al. 2007).

The fact that loss aversion and optimism might optimally compensate for one another has also been pointed out by Yao and Li (2009). They argue that investors who exhibit both anomalies might be most successful (provided that information is sufficiently incomplete).

[11] Several empirical studies have concluded that, to compensate for a loss of any given size, individuals require a gain of roughly twice that size (see e.g. Tom et al. 2007). With the function $v(z) = \beta z^\alpha$, this implies $\gamma = 2^\alpha$ for the parameter of loss aversion, which we approximate by 3/2 for the case when $\alpha = 1/2$.

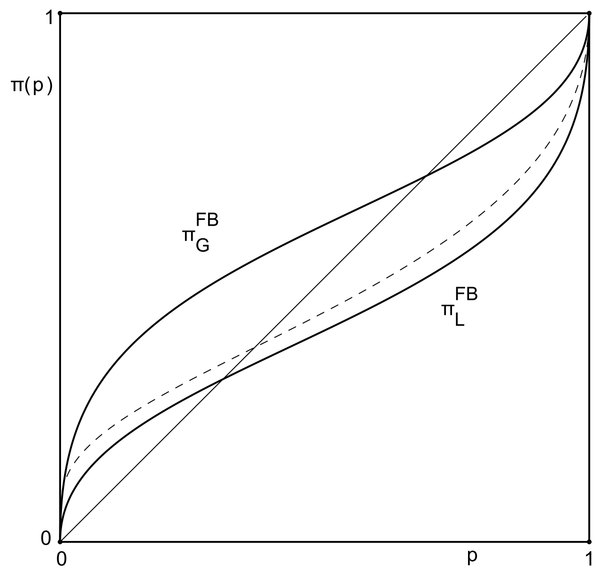

Figure 3: Non-reflective Weighting for $\alpha = 1/2$ and $\gamma = 3/2$.

From the perspective of our model, this provides the basis for the following thought. We have already discussed the evolutionary benefits of reflective probability weighting. Hence it might make sense to impose reflectivity, captured by condition (6), as an additional constraint on evolution's optimization problem. The resulting "third-best" could then involve systematic optimism as an additional compensating anomaly, and would exhibit reflective and inverse S-shaped probability weighting at the same time.[12]

3.2 More General Prospects

In this subsection we want to investigate whether our results apply more generally than just to simple prospects. The use of simple prospects made the analysis tractable and helped to develop an intuition, but it poses the obvious question to what extent the previous results are robust. Hence assume that a prospect consists of a vector of payoffs $\mathbf{z} \in \mathbb{R}^n$, where $n \geq 2$

[12]Several other papers have provided purpose-oriented explanations for biases such as optimism or overconfidence. In Bernardo and Welch (2001), overconfidence helps to solve problems of herding. Compte and Postlewaite (2004) model a situation where some overconfidence is optimal because it increases actual success probabilities. In Carrillo and Mariotti (2000), Bénabou and Tirole (2002) and Brocas and Carrillo (2004), individuals may choose to adhere to delusions because they serve as commitment devices in the presence of time-inconsistency problems. These latter contributions are therefore closer to our approach, by modelling incorrect perceptions of reality as a compensation for pre-existing biases (present bias in these models). Heifetz et al. (2007) study the evolution of general perception biases in strategic settings. In a strategic setting, such biases may be evolutionarily beneficial because they can change the equilibrium structure. Finally, in Johnson and Fowler (2011) overconfidence arises in a hawk-dove type of interaction, based on the assumption that players behave aggressively whenever they believe to be more able than their opponent.

and $\mathbf{z} = (z_1, ..., z_n)$ with $z_1 < ... < z_k < 0 < z_{k+1} < ... < z_n$ for $1 \leq k < n$, so that there are k possible losses and $n - k$ possible gains. The associated probabilities are given by $\mathbf{p} \in]0, 1[^n$, where $\mathbf{p} = (p_1, ..., p_n)$ with $\sum_{i=1}^n p_i = 1$. The model examined previously is included in this setup as the special case where $n = 2$ and $k = 1$. As before, let $\mathscr{P}$ denote the set of all such prospects (for fixed values of n and k, which are suppressed for notational convenience). The usual fitness mapping $F : \mathscr{P} \to \mathbb{R}$ is given by $F(\mathbf{p}, \mathbf{z}) = \mathbf{p} \cdot \mathbf{z}$, where "$\cdot$" represents vector multiplication. The sets $\mathscr{P}^+$ and $\mathscr{P}^-$ of positive and negative fitness lotteries are then defined as before. Now let $\boldsymbol{\pi} : \mathscr{P} \to]0, 1[^n$ be a weighting function that assigns decision weights $\boldsymbol{\pi}(\mathbf{p}, \mathbf{z}) = (\pi_1(\mathbf{p}, \mathbf{z}), ..., \pi_n(\mathbf{p}, \mathbf{z}))$ to prospects, such that $U_{\boldsymbol{\pi}}(\mathbf{p}, \mathbf{z}) = \boldsymbol{\pi}(\mathbf{p}, \mathbf{z}) \cdot V(\mathbf{z})$ is the agent's perceived utility from prospect $(\mathbf{p}, \mathbf{z})$, where $V(\mathbf{z}) \in \mathbb{R}^n$ denotes the vector obtained by mapping $\mathbf{z}$ pointwise through the value function V. The assumptions imposed on V and on h are identical to those described in Section 2.1. Based on the utility function we immediately obtain the sets $\mathscr{P}^+_{\boldsymbol{\pi}}$ and $\mathscr{P}^-_{\boldsymbol{\pi}}$ of prospects that are accepted and rejected, respectively, and the optimization problem we are interested in is given by

$$\min_{\boldsymbol{\pi} \in \mathscr{W}} \int\limits_{\left(\mathscr{P}^-_{\boldsymbol{\pi}} \cap \mathscr{P}^+\right) \cup \left(\mathscr{P}^+_{\boldsymbol{\pi}} \cap \mathscr{P}^-\right)} |F(\mathbf{p}, \mathbf{z})| h(\mathbf{p}, \mathbf{z}) d(\mathbf{p}, \mathbf{z}), \tag{12}$$

for some strictly positive density h on $\mathscr{P}$ and a set of admissible weighting functions $\mathscr{W}$.

Let $\mathscr{F}$ denote the unrestricted set of weighting functions, and $\mathscr{F}'$ the set of functions which depend on $\mathbf{p}$ only. We can now prove the following result.

Proposition 6. *(i) Suppose $n > 2$ and $\mathscr{W} = \mathscr{F}$. Then, the first-best is achievable.*
(ii) Suppose $n > 2$ and $\mathscr{W} = \mathscr{F}'$. Then, the first-best is not achievable.

Proof. Part (i). Let $\boldsymbol{\pi}^{FB}(\mathbf{p}, \mathbf{z})$ be given by $\pi_i^{FB}(\mathbf{p}, \mathbf{z}) = p_i z_i / V(z_i)$, $\forall i = 1, ..., n$. Then we have $\boldsymbol{\pi}^{FB}(\mathbf{p}, \mathbf{z}) \cdot V(\mathbf{z}) = \sum_{i=1}^n \pi_i^{FB}(\mathbf{p}, \mathbf{z}) \cdot V(z_i) = \sum_{i=1}^n p_i z_i = \mathbf{p} \cdot \mathbf{z}$, so that utility and fitness maximization becomes identical and the first-best is achieved.
Part (ii). Suppose $n > 2$ and $\mathscr{W} = \mathscr{F}'$. Assume also that $k \geq 2$ (the proof is analogous when $k = 1$ but $n - k \geq 2$), and let $\boldsymbol{\pi}^* \in \mathscr{F}'$ be a solution to (12). Consider any $(\mathbf{p}, \mathbf{z}^1)$ such that $\mathbf{p} \cdot \mathbf{z}^1 = 0$, and construct $\mathbf{z}^2(\epsilon)$ from $\mathbf{z}^1$ by letting $z_1^2(\epsilon) = z_1^1 - \epsilon$, $z_2^2(\epsilon) = z_2^1 + (p_1/p_2)\epsilon$, and $z_i^2(\epsilon) = z_i^1$ for all $i = 3, ..., n$. Whenever $\epsilon \in]\underline{\epsilon}, \overline{\epsilon}[$, where $\underline{\epsilon} \equiv p_2(z_1^1 - z_2^1)/(p_1 + p_2) < 0$ and $\overline{\epsilon} \equiv \min\{p_2(-z_2^1)/p_1, p_2(z_3^1 - z_2^1)/p_1\} > 0$, we have that $z_1^2(\epsilon) < z_2^2(\epsilon) < \min\{z_3^2(\epsilon), 0\}$, so that $\mathbf{z}^2(\epsilon)$ is well-defined. Furthermore, $\mathbf{p} \cdot \mathbf{z}^2(\epsilon) = 0$ holds for all ϵ. We now obtain

$$\frac{\partial^2 \left[\boldsymbol{\pi}^*(\mathbf{p}) \cdot V(\mathbf{z}^2(\epsilon))\right]}{\partial \epsilon^2} = \pi_1^*(\mathbf{p}) V''(z_1^2(\epsilon)) + \pi_2^*(\mathbf{p}) \left(\frac{p_1}{p_2}\right)^2 V''(z_2^2(\epsilon)) > 0$$

due to strict convexity of V below zero, so that utility, as opposed to fitness, is not constant in $\underline{\epsilon} < \epsilon < \overline{\epsilon}$. Continuity of V then implies that for (every) $\mathbf{p}$ there is a positive measure set of payoffs $\mathbf{z}$ for which $\mathbf{p} \cdot \mathbf{z}$ and $\boldsymbol{\pi}^*(\mathbf{p}) \cdot V(\mathbf{z})$ have the opposite sign, which implies that $\boldsymbol{\pi}^*$ is not first-best. □

The fact that payoff-dependent weighting facilitates first-best behavior is unchanged by the introduction of more general prospects. A change occurs for payoff-independent weighting: the first-best can never be achieved with more than only one gain and loss, not even for specific value functions such as $v(z) = \beta z^\alpha$. It is thus important to keep in mind that all our following results are about actual second-best weighting.

In general, the model becomes hard to tackle analytically for $n > 2$. Therefore, we will proceed to present numerical results for several interesting cases of payoff-independent weighting. Our numerical procedure is described in detail in Appendix A.6. All following results are for the case of two gains and two losses ($n = 4, k = 2$). We use $v_G(z) = z^\alpha$ and $v_L(z) = \gamma z^\alpha$ for the value function, and assume that nature offers prospects according to a uniform prior.[13] For each combination of $\alpha \in \{1, 3/4, 1/2, 1/4, 1/10\}$ and $\gamma \in \{1, 3/2\}$, we search for the optimal weighting function within the *rank-dependent* framework (Quiggin 1982). There, the additional restriction is imposed that there exists a (bijective) function $w : [0,1] \to [0,1]$ which transforms cumulated probabilities. Formally,

$$\pi_i(\mathbf{p}) = w\left(\sum_{j=1}^{i} p_i\right) - w\left(\sum_{j=1}^{i-1} p_i\right). \tag{13}$$

As a result, the obtained decision weights π_i are in fact probabilities because they sum up to one. Cumulative prospect theory (Tversky and Kahneman 1992) uses a similar formulation, where an inverse of cumulated probabilities is transformed for gains.

Karmarkar	$\gamma = 1$		$\gamma = 3/2$	
α	β^*	Fitness	β^*	Fitness
1	1.0000	0.2714	1.3819	0.2646
3/4	0.7708	0.2713	1.2222	0.2612
1/2	0.5208	0.2707	1.2847	0.2555
1/4	0.2639	0.2690	1.2847	0.2467
1/10	0.1042	0.2672	1.1111	0.2433

Table 1: Results for $w(p) = p^\beta/(p^\beta + (1-p)^\beta)$.

[13] Our results appear to be robust to the use of different priors. Appendix A.6 contains all results of this section for a different prior, where payoffs and probabilities are not independent.

Table 1 contains the results when we parameterize $w(p) = p^\beta/(p^\beta + (1-p)^\beta)$ as in Karmarkar (1978, 1979), and search the range $\beta \in [0,2]$ so that linear, S-shape, and inverse S-shape weighting is admitted. The first main column refers to the case without loss aversion ($\gamma = 1$). If we additionally assume that payoffs are perceived linearly ($\alpha = 1$), the optimum does not involve any probability weighting ($\beta^* = 1$). The corresponding behavior is first-best, yielding the largest possible fitness level (0.2714). We now introduce S-shaped payoff valuation by decreasing α towards zero. As Table 1 shows, the optimal exponent of the weighting function does then also go to zero, which means that probability weighting should become increasingly inverse S-shaped, as in our simpler model in Section 2. It is even true that the optimal exponent β^* of probability weighting is almost equal to the exponent α of the value function, which we identified as a property of first-best weighting for simple prospects in Proposition 3. Note again, however, that the first-best can no longer be achieved in the present setup when $\alpha < 1$. As can be seen from Table 1, the optimal fitness level goes down (albeit not too much) as we decrease α, i.e. as we increase the severity of the original distortion.

With loss aversion ($\gamma = 3/2$), the picture looks different. According to the results in Table 1, optimal weighting should now be S-shaped ($\beta^* > 1$), in contrast to both our previous findings and empirical evidence. Before taking this result serious, though, we need to point out that it holds for imposed reflectivity and symmetry, while the model with simple prospects implied that both these properties are violated in an unrestricted optimum under loss aversion (Proposition 5). Let us therefore relax the assumption of symmetry, by considering the more general linear in log odds function $w(p) = \delta p^\beta/(\delta p^\beta + (1-p)^\beta)$, for values $\beta, \delta \in [0,2]$.

LLO	$\gamma = 1$			$\gamma = 3/2$		
α	β^*	δ^*	Fitness	β^*	δ^*	Fitness
1	1.0000	1.0000	0.2714	1.0069	0.6944	0.2713
3/4	0.7708	1.0000	0.2713	0.7708	0.6875	0.2712
1/2	0.5208	1.0000	0.2707	0.5208	0.6806	0.2705
1/4	0.2639	1.0000	0.2690	0.2569	0.6736	0.2688
1/10	0.1042	1.0000	0.2672	0.1042	0.6667	0.2669

Table 2: Results for $w(p) = \delta p^\beta/(\delta p^\beta + (1-p)^\beta)$.

Without loss aversion, the optimum is again achieved by symmetric weighting ($\delta^* = 1$), so that column 1 in Table 2 is otherwise identical to Table 1. A crucial difference arises under loss aversion: allowing for asymmetry restores the inverse S-shape of the weighting function. Also, the emerging asymmetry is strikingly close to empirical findings, with the point of correct probability perception strictly below 1/2. For the case where $\alpha = 1/2$ and $\gamma = 3/2$, the optimally adjusted linear in log odds function is depicted as a solid line in Figure 4. Picking up on our remarks in Section 3.1, we believe that this good empirical fit provides support for the idea of treating reflectivity (but not symmetry) as an additional constraint on the fitness maximization problem. If gain and loss probabilities have to be treated identically, for instance due to large fixed costs of a weighting mechanism, loss aversion implies a systematic deviation from symmetry, in the direction found in many empirical studies.

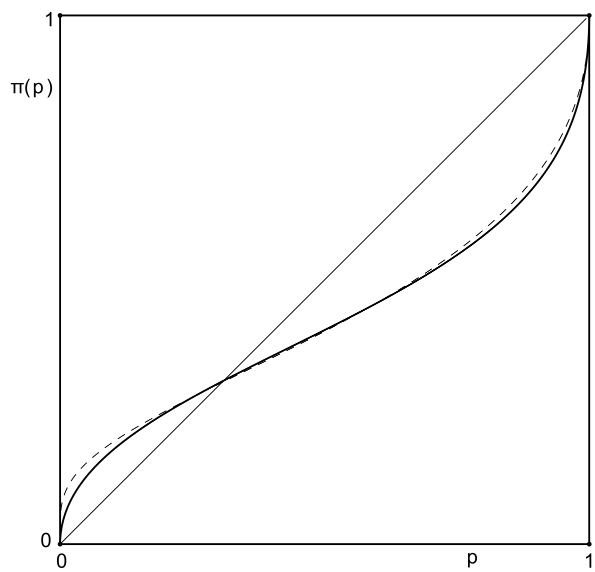

Figure 4: Numerical results for $\alpha = 1/2$ and $\gamma = 3/2$.

The last functional form we consider is Prelec's (1998) generalized function $w(p) = e^{-\delta(-\ln p)^{\beta}}$. Table 3 reveals that optimal weighting is again inverse S-shaped, with a strong correlation between the inverse S-shape of the weighting function (as captured by β^*) and the S-shape of the value function (as captured by α). For $\delta = 1$, the Prelec function is not symmetric whenever $\alpha < 1$. When there is no loss aversion (column 1 in Table 3), this is compensated by letting $\delta^* < 1$, so that the optimal function is at least close to symmetric. With loss aversion, symmetry is again violated systematically. The generalized Prelec function which is optimal for $\alpha = 1/2$ and $\gamma = 3/2$ is depicted by a dotted line in Figure 4.

To resolve doubts that our findings might still depend on the functional forms chosen, we conducted a non-parameterized grid-search for the optimal weighting function $w(p)$ within the rank-dependent framework. The result, contained in Appendix A.6, again confirms that optimal weighting exhibits an inverse S-shape with overweighting of small and underweight-

Prelec	$\gamma = 1$			$\gamma = 3/2$		
α	β^*	δ^*	Fitness	β^*	δ^*	Fitness
1	1.0000	1.0000	0.2714	0.9306	1.2431	0.2713
3/4	0.7708	0.9306	0.2713	0.7014	1.1597	0.2712
1/2	0.5347	0.8542	0.2706	0.4931	1.0903	0.2705
1/4	0.2708	0.7708	0.2688	0.2569	1.0069	0.2688
1/10	0.1111	0.7222	0.2668	0.1042	0.9514	0.2669

Table 3: Results for $w(p) = e^{-\delta(-\ln p)^\beta}$.

ing of large probabilities.

4 Discussion

The main insight of our analysis is that non-linear probability weighting is a complement of non-linear payoff valuation. As a consequence, probability weighting should be more pronounced for agents whose value function deviates more strongly from linearity. This can be seen most easily for the case of first-best weighting, where the exponent of the optimal linear in log odds function is equal to the value function's exponent, and it has further been illustrated in the previous Section 3.2. We now want to outline how that insight could become the basis for empirical testing. Some obstacles have to be addressed, however, before the above described correlation translates into a testable prediction.

1. *Mutual optimality.* We have derived optimal weighting for given value functions. Long-run evolutionary stability might also require inverse optimality of the value function for the given weighting function.

2. *Polymorphic populations.* When there are different optimal value-weighting-function pairs, it is not evident that evolution would support a polymorphic equilibrium in which several of them are present.

Concerning mutual optimality, observe that it follows immediately for all the cases where the first-best is achievable. When simple prospects are concerned and probabilities are weighted by $\pi(p) = p^\alpha/(p^\alpha + (1-p)^\alpha)$ for some α, then $v(z) = z^\alpha$ is indeed optimal because the best possible fitness level is attained. Although this result is not yet entirely satisfactory, it at least serves as an illustration that different, mutually optimal pairs of weighting and value functions can indeed exist. On the other hand, one can also think of evolutionary dynamics or stability concepts that do not require mutual optimality. For instance, staggered development might imply that evolutionarily younger attributes are chosen optimal given the existing attributes, so that inverse optimality does not follow necessarily in this case.

The problem with polymorphic populations is twofold. First, coexistence of different types in a model with multiple traits either requires that reproduction is asexual (as in Ely 2009), or that probability weighting function and value function are inherited jointly, since otherwise genetic recombination systematically yields mismatches (Waldman 1994). Alternatively, the problem could be overcome by assortativity within a heterogeneous population.[14] Second, coexistence of different value-weighting-function pairs would require them to achieve exactly the same fitness level, while our results in Section 3.2 indicate that fitness goes down as the value function's S-shape becomes more pronounced, even with optimally adapted weighting function. On the other hand, this decline is small: in Table 2, for instance, the fitness level achievable with a value function exponent of $\alpha = 1/4$ is still more than 99% of the first-best fitness, with and without loss-aversion. Hence an appropriately defined evolutionary ϵ-equilibrium could easily comprise heterogeneity.

We are not aware of a study that either explicitly examines the correlation between value and weighting function curvatures, or provides enough individual-level estimation results to make meaningful correlation statements. Gonzalez and Wu (1999) present estimated parameters for only 10 subjects, Hsu et al. (2009) for only 16 subjects.[15] Qiu and Steiger (2010) estimate value and weighting functions for 124 subjects, and conclude that there is no positive correlation. Their measure of the probability weighting function is, however, the relative area below the function, which captures elevation rather than curvature.[16] More helpful for our purpose is the contribution by Bruhin et al. (2010). Based on experimental data for altogether 448 subjects, they estimate a finite mixture model in which individuals are endogenously grouped into different behavioral categories. They find that (only) two

[14] We are greatful to Joel Sobel for pointing out this possibility.

[15] Gonzalez and Wu (1999) consider gain lotteries only, and perform nonparametric estimates of individual value and weighting functions. Then they show that the nonparametric results can be approximated by the CRRA value function and the linear in log odds weighting function. Using their estimates (Table 3, p. 157), we obtain a value of -0.05 for the coefficient of correlation between the estimated exponents of the value and the weighting function. Statements about significance are not straightforward because the parameters are themselves only estimated. When treating them as a standard sample, however, this result is not significantly different from zero at any reasonable level. Hsu et al. (2009) are mostly interested in the neural correlates of probability weighting, but also present estimated parameters from a behavioral experiment. They also consider gain lotteries only, and fit a CRRA value function together with the one parameter Prelec function (10). Based on their estimates (Table S4, p. 13 online supplementary material), we obtain a coefficient of correlation of $+0.04$, which is again insignificant.

[16] Qiu and Steiger (2010) also estimate a linear in log odds function, but they provide a (nonsignificant) coefficient of correlation only between the exponent α of CRRA utility and the parameter δ of the weighting function, rather than the exponent β, in which we would be interested. Note that Qiu and Steiger (2010) consider only gains (no losses are possible), and the weighting functions they estimate are mostly not inverse S-shaped but convex.

such categories emerge: 20% of all individuals maximize expected payoffs, using linear value and probability weighting functions, while 80% exhibit inverse S-shape probability weighting. From our perspective, these types could correspond to two different points from the continuum of second-best possibilities. However, Bruhin et al. (2010) also find that the value function of the group that weighs probabilities non-linearly is concave for gains but not convex for losses (Table 5, p. 25). This finding is at odds with our basic premise of an S-shaped value function, and makes it impossible to directly test our model.

5 Conclusions

Substantial progress has been made in the theory of bounded rationality during the last decades. There is, however, still widespread unease about relaxing the assumption of rationality, often based on the argument that behavioral economics offers many different explanations for different phenomena, but not a unifying model. For instance, Rubinstein (1998) describes a view according to which "...rationality of the decision maker can be seen as the minimal discipline to be imposed on the *modeler* [emphasis added]. Our departure from the rational man paradigm represents a removal of those chains. However, there are an infinite number of "plausible" models that can explain social phenomena; without such chains we are left with a strong sense of arbitrariness" (p. 4).

In this paper, we have shown that the probability weighting function is in fact not arbitrary but complements the adaptive S-shaped payoff valuation function in a second-best sense. The underlying idea is more general: behavioral anomalies should not be examined in isolation, but must be understood as an interacting system. Once cognitive constraints introduce an anomaly, we should expect deviations from perfectly rational behavior elsewhere, too. Recently, a couple of papers have started to apply this insight within an evolutionary context. We believe that this approach might provide a generalized framework for thinking about bounded rationality. Models based on the evolutionary second-best idea are able to make predictions about the co-occurrence or mutual exclusion of behavioral anomalies, and as such could help to organize the multiplicity of empirically documented behavioral phenomena.

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A Appendix

A.1 The Transformed Objective

The objective from (3) can be rewritten as

$$\int_{\mathscr{P}_\eta^+\cap\mathscr{P}^+} F(q,x,y)h(q,x,y)d(q,x,y) + \int_{\mathscr{P}_\eta^+\cap\mathscr{P}^-} F(q,x,y)h(q,x,y)d(q,x,y)$$

because $\mathscr{P}^+$ and $\mathscr{P}^-$ form a partition of $\mathscr{P}$. This in turn is equivalent to

$$\begin{aligned}\int_{\mathscr{P}_\eta^+\cap\mathscr{P}^+} F(q,x,y)h(q,x,y)d(q,x,y) &+ \int_{\mathscr{P}_\eta^+\cap\mathscr{P}^-} F(q,x,y)h(q,x,y)d(q,x,y)\\ + \int_{\mathscr{P}_\eta^-\cap\mathscr{P}^+} F(q,x,y)h(q,x,y)d(q,x,y) &- \int_{\mathscr{P}_\eta^-\cap\mathscr{P}^+} F(q,x,y)h(q,x,y)d(q,x,y),\end{aligned}$$

where the same term has been added and subtracted. Since $\mathscr{P}_\eta^+$ and $\mathscr{P}_\eta^-$ also form a partition of $\mathscr{P}$, we can rearrange to

$$\begin{aligned}&\int_{\mathscr{P}^+} F(q,x,y)h(q,x,y)d(q,x,y)\\ &\quad-\left(\int_{\mathscr{P}_\eta^-\cap\mathscr{P}^+} F(q,x,y)h(q,x,y)d(q,x,y) - \int_{\mathscr{P}_\eta^+\cap\mathscr{P}^-} F(q,x,y)h(q,x,y)d(q,x,y)\right).\end{aligned}$$

Since the first term in this expression is independent of η, we can equivalently minimize the expression in brackets. Observing that $F(q,x,y)\geq 0$ for all $(q,x,y)\in\mathscr{P}_\eta^-\cap\mathscr{P}^+$ and $F(q,x,y)<0$ for all $(q,x,y)\in\mathscr{P}_\eta^+\cap\mathscr{P}^-$, this immediately yields the transformed problem (4).

A.2 Proof of Proposition 2

We first rewrite problem (4) for the present case. We write v for the identical functions v_G and v_L, and we use the fact that $\mathscr{W}=\mathscr{F}'$. Then, for any $q\in\mathbb{R}_+$, define $\mathscr{X}^+(q)=\{(x,y)\in\mathbb{R}_+^2|q\geq y/x\}$ and $\mathscr{X}^-(q)=\{(x,y)\in\mathbb{R}_+^2|q<y/x\}$. For any $r\in\mathbb{R}_+$ let $\mathscr{V}^+(r)=\{(x,y)\in\mathbb{R}_+^2|r\geq v(y)/v(x)\}$

and $\mathscr{V}^-(r) = \{(x,y) \in \mathbb{R}^2_+ | r < v(y)/v(x)\}$. Then, we can reformulate (4) as follows:

$$\min_{\eta \in \mathscr{F}'} \int_{\mathbb{R}_+} \left(\int_{(\mathscr{V}^-(\eta(q)) \cap \mathscr{X}^+(q)) \cup (\mathscr{V}^+(\eta(q)) \cap \mathscr{X}^-(q))} |F(q,x,y)| h(q,x,y) d(x,y) \right) dq. \tag{14}$$

For any fixed value of q, the inner integral in (14) is affected by the function η only through $\mathscr{V}^+(\eta(q))$ and $\mathscr{V}^-(\eta(q))$, where in turn only the function value $\eta(q)$ is relevant. Hence $\eta^* \in \mathscr{F}'$ is a solution to (14) if and only if

$$\eta^*(q) \in \arg\min_{r \in \mathbb{R}_+} \int_{\mathscr{V}^-(r) \cap \mathscr{X}^+(q)} |F(q,x,y)| h(q,x,y) d(x,y) + \int_{\mathscr{V}^+(r) \cap \mathscr{X}^-(q)} |F(q,x,y)| h(q,x,y) d(x,y) \tag{15}$$

for almost all $q \in \mathscr{R}_+$. Denote the first integral in (15) by $\Lambda_1(q,r)$, the second by $\Lambda_2(q,r)$, and let $\Lambda(q,r) = \Lambda_1(q,r) + \Lambda_2(q,r)$, so that $\eta^*(q) \in \arg\max_{r \in \mathbb{R}_+} \Lambda(q,r)$ must hold for almost all $q \in \mathbb{R}_+$.

Fix any $q < 1$. We prove the following preliminary claim.

Lemma 2. *Given any $q < 1$, $\Lambda_2(q,r) = 0$ if $r \leq q$.*

Proof. If $r \leq q$ holds, any $(x,y) \in \mathscr{V}^+(r)$ satisfies

$$\frac{v(y)}{v(x)} \leq r \leq q < 1,$$

where the first inequality follows from the definition of $\mathscr{V}^+(r)$. Hence $v(y) < v(x)$ and thus $y < x$, so that strict concavity of v (together with $v(0) = 0$) implies

$$\frac{y}{x} < \frac{v(y)}{v(x)}$$

for all $(x,y) \in \mathscr{V}^+(r)$. But then $y/x < q$, which implies $(x,y) \notin \mathscr{X}^-(q)$. Hence $\mathscr{V}^+(r) \cap \mathscr{X}^-(q) = \varnothing$, so that $\Lambda_2(q,r) = 0$. $\square$

Hence the mistake of accepting a negative fitness prospect does not occur for $q < 1$ when $\eta(q) \leq q$. Now consider the other type of mistake. $\mathscr{V}^-(r)$ contains all payoffs (x,y) for which $r < v(y)/v(x)$. Hence $\mathscr{V}^-(r)$ and therefore $\mathscr{V}^-(r) \cap \mathscr{X}^+(q)$ is weakly shrinking (in the sense of set inclusion) in r, so that $\Lambda_1(q,r)$ is weakly decreasing in r. Combined with Lemma 2 this implies that, whenever $q < 1$, $\Lambda(q,r)$ is weakly decreasing in r up to and including q.

We complete the proof for $q < 1$ by showing that $\Lambda(q,r)$ is strictly decreasing in r at $r = q$, which implies that any solution η^* must satisfy $\eta^*(q) > q$. Let $r = q + \epsilon$, for $\epsilon \geq 0$ with $\epsilon < 1 - q$. Then, any $(x,y) \in (\mathscr{V}^+(q+\epsilon) \cap \mathscr{X}^-(q))$ must satisfy

$$y \in \,]qx, (q+\epsilon)x[\,,$$

where the lower bound follows from the definition of $\mathscr{X}^-(q)$. To derive the upper bound, note that $(x,y) \in \mathscr{V}^+(q+\epsilon)$ implies $v(y)/v(x) \leq q+\epsilon < 1$, which implies $y < x$ and hence $y/x < v(y)/v(x) \leq q+\epsilon$ by concavity of v. Hence we have

$$\Lambda_2(q, q+\epsilon) \leq \int\limits_{\mathbb{R}_+} \left(\int\limits_{qx}^{(q+\epsilon)x} |F(q,x,y)| h(q,x,y) dy \right) dx \equiv \tilde{\Lambda}_2(q,\epsilon), \tag{16}$$

i.e. $\tilde{\Lambda}_2(q,\epsilon)$ is an upper bound for $\Lambda_2(q, q+\epsilon)$. Also, it holds that

$$\frac{\partial \tilde{\Lambda}_2(q,\epsilon)}{\partial \epsilon} = \int\limits_{\mathbb{R}_+} x |F(q,x,(q+\epsilon)x)| h(q,x,(q+\epsilon)x)\, dx.$$

Then $\partial \tilde{\Lambda}_2(q,0)/\partial\epsilon = 0$ because $F(q,x,qx) = 0$. Observe that $\tilde{\Lambda}_2(q,0) = 0$, and $\Lambda_2(q,q+0) = 0$ by Lemma 2, so that $\tilde{\Lambda}_2(q,\epsilon)$ and $\Lambda_2(q,q+\epsilon)$ coincide for $\epsilon = 0$. Altogether, this implies that $\partial\Lambda_2(q,q+\epsilon)/\partial\epsilon = 0$ at $\epsilon = 0$ must also hold. Now consider $\Lambda_1(q,q+\epsilon)$. A tuple (x,y) is an element of $\mathscr{V}^-(q+\epsilon) \cap \mathscr{X}^+(q)$ if and only if

$$y \in \left] v^{-1}\left[(q+\epsilon)v(x)\right], qx \right],$$

where v^{-1} is the inverse of v, and $v^{-1}\left[(q+\epsilon)v(x)\right] < qx$ for sufficiently small ϵ due to concavity of v and $q < 1$. Hence

$$\Lambda_1(q,q+\epsilon) = \int\limits_{\mathbb{R}_+} \left(\int\limits_{v^{-1}[(q+\epsilon)v(x)]}^{qx} |F(q,x,y)| h(q,x,y) dy \right) dx$$

such that

$$\frac{\partial \Lambda_1(q,q+\epsilon)}{\partial\epsilon} = \\ -\int\limits_{\mathbb{R}_+} \left(\frac{v(x)}{v'\left[v^{-1}\left[(q+\epsilon)v(x)\right]\right]} \right) \left|F(q,x,v^{-1}\left[(q+\epsilon)v(x)\right])\right| h(q,x,v^{-1}\left[(q+\epsilon)v(x)\right])\, dx.$$

From the fact that $v^{-1}\left[qv(x)\right] < qx$ it then immediately follows that $\partial\Lambda_1(q,q+\epsilon)/\partial\epsilon < 0$ at $\epsilon = 0$. Hence we have

$$\left.\frac{\partial\Lambda(q,q+\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} = \left.\frac{\partial\Lambda_1(q,q+\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} + \left.\frac{\partial\Lambda_2(q,q+\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} < 0,$$

which completes the proof for $q < 1$.

The arguments for $q = 1$ and $q > 1$ are analogous and therefore omitted.

A.3 Proof of Lemma 1

Part (i). Suppose $\eta \in \mathscr{F}'$ satisfies (6) and consider the reflective candidate $\pi_\eta(p) = \eta(p/(1-p))/(1+\eta(p/(1-p)))$. It implements the relative perception

$$\frac{\pi_\eta\left(\frac{q}{1+q}\right)}{\pi_\eta\left(\frac{1}{1+q}\right)} = \frac{\eta\left(\frac{q}{1+q} \times \frac{1+q}{1}\right)}{1+\eta\left(\frac{q}{1+q} \times \frac{1+q}{1}\right)} \times \frac{1+\eta\left(\frac{1}{1+q} \times \frac{1+q}{q}\right)}{\eta\left(\frac{1}{1+q} \times \frac{1+q}{q}\right)} = \frac{\eta(q)}{1+\eta(q)} \frac{1+\eta(1/q)}{\eta(1/q)}.$$

Using (6) we obtain

$$\frac{\eta(q)}{1+\eta(q)} \frac{1+\eta(1/q)}{\eta(1/q)} = \frac{\eta(q)^2}{1+\eta(q)} \left[\frac{\eta(q)+1}{\eta(q)}\right] = \eta(q),$$

which shows that π_η indeed implements η. For the converse, suppose η is implemented by some reflective weighting π, i.e. $\eta(q) = \pi(q/(1+q)/\pi(1/(1+q))$ for all $q \in \mathbb{R}_+$. Then we have

$$\eta(1/q) = \frac{\pi\left(\frac{1/q}{1+1/q}\right)}{\pi\left(\frac{1}{1+1/q}\right)} = \frac{\pi\left(\frac{1}{1+q}\right)}{\pi\left(\frac{q}{1+q}\right)} = 1/\eta(q)$$

for all $q \in \mathbb{R}_+$, so η satisfies (6).

Part (ii). Suppose η^* satisfies (6) and is implemented by reflective weighting π_{η^*} Then we have by Proposition 2 that

$$\frac{\pi_{\eta^*}(q/(1+q))}{\pi_{\eta^*}(1/(1+q))} \gtreqless q \Leftrightarrow q \lesseqgtr 1.$$

Substituting $q = p/(1-p)$ then immediately yields condition (7). Also, the substitution shows that

$$\eta^*(p/(1-p)) = \frac{\pi_{\eta^*}(p)}{\pi_{\eta^*}(1-p)}.$$

Imposing symmetry $\pi_{\eta^*}(1-p) = 1 - \pi_{\eta^*}(p)$ and rearranging yields the unique solution $\pi_{\eta^*}(p) = \eta^*(p/(1-p))/(1+\eta^*(p/(1-p)))$, and straightforward calculations reveal that it satisfies (8).

A.4 Proof that (10) satisfies condition (7)

Substituting $\pi(p) = e^{-[(-\ln p)^\alpha]}$ into

$$\frac{\pi(p)}{\pi(1-p)} \gtreqless \frac{p}{1-p} \tag{17}$$

and rearranging leads to

$$\left(\ln\left(\frac{1}{1-p}\right)\right)^\alpha - \ln\left(\frac{1}{1-p}\right) + \ln\left(\frac{1}{p}\right) - \left(\ln\left(\frac{1}{p}\right)\right)^\alpha \gtreqless 0. \tag{18}$$

Let $f_\alpha(p)$ denote the left hand side of (18). Note that $f_\alpha(\frac{1}{2}) = 0$. Hence it is sufficient to show that $f_\alpha(p)$ falls strictly in p, or that $f'_\alpha(p) < 0$ for $0 < \alpha < 1$. Consider

$$f'_\alpha(p) = \alpha\left(\frac{(-\ln(1-p))^{\alpha-1}}{1-p} + \frac{(-\ln(p))^{\alpha-1}}{p}\right) - \frac{1}{1-p} - \frac{1}{p}. \tag{19}$$

The value of the derivative is strictly increasing in α and for $\alpha = 1$ is is exactly 0. Hence it must be strictly negative for $\alpha < 1$, what completes the proof.

A.5 Proof of Proposition 4

We prove the proposition by two consecutive lemmas.

Lemma 3. *Suppose $\mathscr{W} = \mathscr{F}'$ and $\gamma \geq 1$. Then, the first-best can be achieved if and only if there exists a function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\frac{\gamma v(qz)}{v(z)} = \delta(q) \tag{20}$$

for all $z \in \mathbb{R}_+$.

Proof. Suppose the first-best is achievable by some function η^{FB}. Problem (4) can again be reformulated to (15), with the only difference that now $\mathscr{V}^+(r) = \{(x,y) \in \mathbb{R}^2_+ | r \geq \gamma v(y)/v(x)\}$ and $\mathscr{V}^-(r) = \{(x,y) \in \mathbb{R}^2_+ | r < \gamma v(y)/v(x)\}$. The solution being first-best implies that $\Lambda(q, \eta^{FB}(q)) = 0$ holds for almost all $q \in \mathbb{R}_+$. This is equivalent to saying that, for almost all q, $\mathscr{V}^-(\eta^{FB}(q)) \cap \mathscr{X}^+(q)$ and $\mathscr{V}^+(\eta^{FB}(q)) \cap \mathscr{X}^-(q)$ are measure-zero sets. Fix some value $q \in \mathbb{R}_+$. The line $y = qx$ separates the positive quadrant $\mathbb{R}^2_+$ into $\mathscr{X}^+(q)$ and $\mathscr{X}^-(q)$. The curve $y = v^{-1}\left[\eta^{FB}(q)v(x)/\gamma\right]$ separates $\mathbb{R}^2_+$ into $\mathscr{V}^+(\eta^{FB}(q))$ and $\mathscr{V}^-(\eta^{FB}(q))$. Thus for almost all $q \in \mathbb{R}_+$, $v^{-1}\left[\eta^{FB}(q)v(x)/\gamma\right] = qx$ holds for almost all $x \in \mathbb{R}_+$, and hence for all $x \in \mathbb{R}_+$ by continuity of v. Equivalently, for almost all $q \in \mathbb{R}_+$,

$$\frac{\gamma v(qx)}{v(x)} = \eta^{FB}(q), \forall x \in \mathbb{R}_+, \tag{21}$$

holds. Continuity of v then implies that there exists a function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$, which coincides with η^{FB} except possibly at countably many points, which satisfies (20).

Conversely, if a function δ exists which satisfies (20), then $\eta^{FB} = \delta$ is a solution which achieves the first-best, from the same arguments. □

Lemma 4. *The following two statements are equivalent.*
(i) $v : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, strictly increasing, strictly concave, and satisfies (20).
(ii) $v : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $v(z) = \beta z^\alpha$ for some $\beta > 0$ and $0 < \alpha < 1$.

Proof. *(ii)* $\Rightarrow$ *(i)*. This implication is obvious, with $\delta(q) = \gamma q^\alpha$ being the function required in (20).

$(i) \Rightarrow (ii)$. Suppose v satisfies (i). Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by $f(z) = v(z)/v(1)$. Clearly, f still satisfies (i). Specifically, the function $\tilde{\delta}(z) \equiv \delta(z)/\gamma$ satisfies $f(qz) = \tilde{\delta}(q)f(z)$ for all $z, q \in \mathbb{R}_+$. Setting $z = 1$ we obtain $f(q) = \tilde{\delta}(q)$, for all $q \in \mathbb{R}_+$. Multiplication of both sides by $f(z)$ then yields that $f(q)f(z) = \tilde{\delta}(q)f(z)$ for all $q, z \in \mathbb{R}_+$. But since $\tilde{\delta}(q)f(z) = f(qz)$ from above, we obtain that $f(q)f(z) = f(qz)$ for all $q, z \in \mathbb{R}_+$, i.e. the function f is totally multiplicative. Since it is also continuous and strictly increasing, it follows that f must be of the form $f(z) = z^\alpha$ for some $\alpha > 0$ (see Cohen (1989), p. 513). Since f is strictly concave, $\alpha < 1$ must also hold. Then, $v(z) = v(1)f(z)$ implies that $v(z) = \beta z^\alpha$ for some $\beta > 0$ and $0 < \alpha < 1$. □

Lemmas 3 and 4 together imply that, when $\mathscr{W} = \mathscr{F}'$, the first-best can be achieved if and only if the function v (which is continuous, strictly increasing, and strictly concave) is given by $v(z) = \beta z^\alpha$ for some $\beta > 0$ and $0 < \alpha < 1$. The fact that $\eta^{FB}(q) = \gamma q^\alpha$ is a first-best solution then follows immediately from the proof of Lemma 3.

A.6 Numerical Procedure

All numerical calculations were performed in GNU Octave. The scripts are available upon request.

Throughout, we consider the case with two losses and two gains. We start with defining a finite version of the set of prospects $\mathscr{P}$. First, we discretize the interval $[0, 1]$ of probabilities into a grid of size n_p, i.e. we allow for probabilities $0, 1/n_p, 2/n_p, ..., 1$. We can then generate the set of all probability vectors $\mathbf{p} = (p_1, p_2, p_3, p_4)$ based on this grid. Analogously, we allow for payoffs between $-k$ and $+k$ and discretize $[-k, k]$ into a grid of size n_z, so that the set of all payoff vectors $\mathbf{z} = (z_1, z_2, z_3, z_4)$ as defined in Section 3.2 can be derived for that grid. The set $\mathscr{P}$ of all prospects is then made up of all possible combinations of probabilities and payoffs, and nature's prior is an assignment of probabilities to each of these prospects. We use $n_p = n_z = 10$ and $k = 2$, except if stated otherwise.

The value function is given by $v_G(z) = z^\alpha$ and $v_L(z) = \gamma z^\alpha$, and we repeat the following procedure for each combination of $\alpha \in \{1, 3/4, 1/2, 1/4, 1/10\}$ and $\gamma \in \{1, 3/2\}$. For any of the considered weighting functions (Karmarkar, linear in log odds, Prelec), we search the range $[0, 2]$ for the optimal parameter values, i.e. the values of the weighting functions' parameters that maximize expected fitness. The search is carried out in two stages. First, we decompose $[0, 2]$ into a grid of size n_g and identify the optimum on that grid. As a second step, we search the area around this optimum ($\pm 1/n_g$) more closely, by again decomposing it into a grid of size n_g. We use $n_g = 12$ throughout.

For the nonparametric approach, we go through all increasing functions $\{0, 1/n_p, 2/n_p, ..., 1\} \to \{0, 1/n_p, 2/n_p, ..., 1\}$, calculate the associated fitness and identify the optimum. We have used $n_p = 12$ here. For $\alpha = 1/2$ and $\gamma = 1$, the result is displayed in Figure 5.

All results in Section 3.2 are based on a uniform prior, where all prospects from $\mathscr{P}$ are weighted equally. As a robustness check, we have performed the same calculations for a different prior.

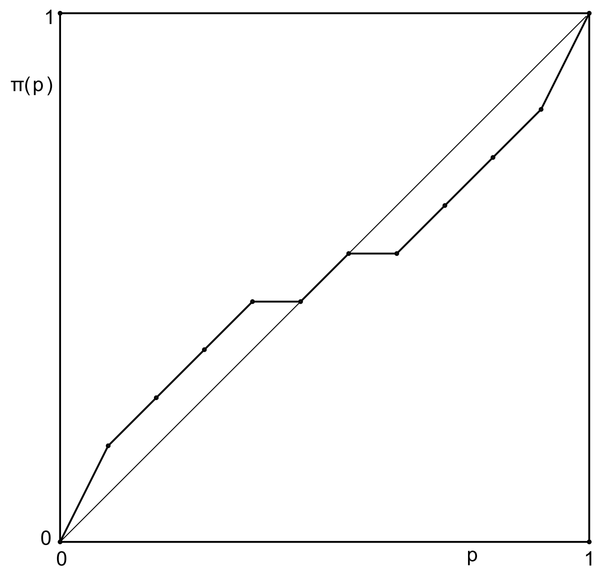

Figure 5: Nonparametric Result

Specifically, we consider the case where the likelihood of a prospect $(\mathbf{p}, \mathbf{z})$ is inversely related to its absolute fitness value $|\mathbf{p} \cdot \mathbf{z}|$.[17] On the one hand this captures a realistic scenario where more extreme prospects are less common than more moderate prospects. On the other hand, probabilities and payoffs are no longer statistically independent with this prior, unlike for the uniform case. The corresponding results (Tables 4 to 6) appear below in the same order as Tables 1 to 3 in Section 3.2. It can be seen that there are only very small, if any, differences in the optimal parameters.

Karmarkar	$\gamma = 1$		$\gamma = 3/2$	
α	β^*	Fitness	β^*	Fitness
1	1.0000	0.1226	1.4097	0.1108
3/4	0.7708	0.1222	1.3403	0.1074
1/2	0.5208	0.1207	1.3611	0.1025
1/4	0.2639	0.1178	1.2569	0.0970
1/10	0.1042	0.1157	1.1042	0.0953

Table 4: $w(p) = p^\beta/(p^\beta + (1-p)^\beta)$, nonuniform prior.

[17]Formally, the probability that nature offers a prospect $(\mathbf{p}, \mathbf{z})$ is assumed to be directly proportional to $1/(|\mathbf{p} \cdot \mathbf{z}| + 0.05)$, where 0.05 is added to deal with zero fitness prospects.

LLO	$\gamma = 1$			$\gamma = 3/2$		
α	β^*	δ^*	Fitness	β^*	δ^*	Fitness
1	1.0000	1.0000	0.1226	1.0069	0.6944	0.1223
3/4	0.7708	1.0000	0.1222	0.7708	0.6875	0.1218
1/2	0.5208	1.0000	0.1207	0.5139	0.6806	0.1202
1/4	0.2639	1.0000	0.1178	0.2569	0.6736	0.1174
1/10	0.1042	1.0000	0.1157	0.1042	0.6667	0.1152

Table 5: $w(p) = \delta p^\beta / (\delta p^\beta + (1-p)^\beta)$, nonuniform prior.

Prelec	$\gamma = 1$			$\gamma = 3/2$		
α	β^*	δ^*	Fitness	β^*	δ^*	Fitness
1	1.0000	1.0000	0.1226	0.9306	1.2431	0.1222
3/4	0.7708	0.9306	0.1221	0.7014	1.1667	0.1218
1/2	0.5278	0.8542	0.1203	0.4931	1.0903	0.1202
1/4	0.2708	0.7708	0.1174	0.2569	1.0069	0.1174
1/10	0.1111	0.7222	0.1150	0.1042	0.9514	0.1152

Table 6: $w(p) = e^{-\delta(-\ln p)^\beta}$, nonuniform prior.