# Efficient Risk Sharing with Limited Commitment and Hidden Storage*

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**Abstract**

We extend the model of risk sharing with limited commitment (e.g. Kocherlakota, 1996) by introducing both a private (non-contractible and/or non-observable) and a public storage technology. We first show that agents' Euler constraints are violated at the constrained-efficient allocation of the basic model under general conditions. We then study a problem where agents' default and saving incentives are both taken into account. We show that when the planner and the agents have access to the same intertemporal technology, agents no longer want to store at the constrained-efficient allocation. The reason is that the planner has more incentive to save, because she internalizes the positive effect of aggregate assets on future risk sharing by relaxing participation constraints. Public storage is positive even when there is no aggregate uncertainty and the return on storage is below the discount rate. This is in contrast to the case where hidden income or effort is the deep friction that limits risk sharing (e.g. Cole and Kocherlakota, 2001). We also show that assets remain stochastic whenever only moderate risk sharing is implementable in the long run, but become constant if high but still imperfect risk sharing is the long-run outcome. However, if the return on the intertemporal technolgy is as high as the discount rate, perfect risk sharing is always self-enforcing in the long run. Further, higher consumption inequality implies higher public asset accumulation. In terms of consumption dynamics, we overturn three counterfactual predictions of the basic limited commitment model. In particular, the amnesia and persistence properties do not hold in our model when assets are stochastic in the long run. Further, agents' Euler inequalities hold by construction.

**Keywords**: risk sharing, limited commitment, hidden storage, dynamic contracts

**JEL codes**: E20

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# 1 Introduction

In the existing models of risk sharing with limited commitment, agents' incentive to deviate from the constrained-efficient allocation by saving is ignored. At the same time, it has been shown that hidden storage reduces risk sharing in environments with private information frictions, see Allen (1985), Cole and Kocherlakota (2001), and Ábrahám, Koehne, and Pavoni (2011). To our knowledge, only observable and contractable saving has been considered in the limited commitment framework, see Ligon, Thomas, and Worrall (2000). In that environment, the optimal allocation is conditioned on individual asset holdings, while this is not possible when savings are hidden.

In several economic contexts where the model of risk sharing with limited commitment has been applied, agents are likely to have a way to save, using some 'backyard' technology or simple storage. In the context of village economies (Ligon, Thomas, and Worrall, 2002), households may keep grain or cash around the house for self-insure purposes. Households in the United States (Krueger and Perri, 2006) may keep savings in cash or 'hide' their assets abroad. Spouses within a household (Mazzocco, 2007) may also keep savings for personal use. In all these examples the amount of private savings is not observable and/or contractible.

This paper extends the literature on risk sharing with limited commitment by introducing a private (non-contractible and/or non-observable) storage technology that agents have access to, as well as a public storage technology. Our starting point is the two-sided lack of commitment framework of Kocherlakota (1996), that we will often refer to as the basic model hereafter. Agents are infinitely lived, risk averse, and ex-ante identical. They receive a risky endowment each period. We assume that there is no aggregate uncertainty. Agents may make transfers to each other in order to smooth their consumption. These transfers are subject to limited commitment, i.e. each agent must be at least as well off as in autarky at each time and state of the world.

The storage technology we introduce allows agents to transfer consumption from one period to the next and earn a net return $r$.[1] Borrowing is not allowed. Access to hidden storage not only changes the value of autarky, but it may also enlarge the set of possible deviations along the equilibrium path. That is, agents could default, or store, or both at the same time or in different periods.

We first study under what conditions agents have an incentive to store at the constrained-efficient allocation of the basic model when storage yields zero interest. We refer to this

[1] The return $r$ can take any value such that $-1 \leq r \leq \frac{1}{\beta} - 1$, where $\beta$ is the subjective discount factor. If $r = -1$, we are back to the basic model. We say that the storage technology is efficient if $r = \frac{1}{\beta} - 1$.

intertemporal technology as pure storage. We show that if income takes at least three values, the possibility of hidden pure storage will affect the constrained-efficient allocation directly, i.e. agents' Euler constraints will be violated, whenever the solution is not 'too close' to full insurance. Afterwards, we show that whenever risk sharing is partial, there exists a threshold return on storage, $\tilde{r} < \frac{1}{\beta} - 1$, above which the Euler constraint binds at the constrained-efficient allocation. This means that whenever the return on private storage is high enough (where some $r \leq 0$ can be high enough), and the basic limited commitment model exhibits relatively little risk sharing, the constrained-efficient allocation in the basic model is not incentive compatible if agents have access to hidden storage. In other words, hidden storage matters under general conditions.[2] The key intuition is that, in the constrained-optimal allocation, agents with high current income and consumption face a decreasing intertemporal pattern of consumption. If the return on storage is high enough, this consumption path cannot be incentive compatible.

Afterwards, we describe our model, where agents' incentive to default on transfers and their incentive to save are both taken into account, unlike in the previous literature. This means that, in addition to the participation/enforceability constraints that make sure that agents will not default, the Euler inequalities of both agents become constraints to rule out deviations by storage, under the first-order condition approach.[3] We allow the social planner to save using the same intertemporal technology as the agents. Without allowing the social planner to store, the feasible set may be empty when agents have access to storage in autarky. Public savings are also relevant for applications. Think of community grain storage facilities in developing countries or the European Financial Stability Facility and the proposed European Stability Mechanism for the euro area. In the rest of this paper, we focus on the constrained-efficient allocations in this environment.

With agents' Euler inequalities as constraints, the temporary Pareto weights (Marcet and Marimon, 2011) or promised utilities (Abreu, Pearce, and Stacchetti, 1990) are no longer sufficient statistics to summarize the past history of income realizations. In models with a dynamic moral hazard problem, hidden storage can be accommodated in a recursive manner using the agent's marginal utility as a co-state variable (Werning, 2001; Ábrahám and Pavoni, 2008). In our environment, this approach would raise serious tractability issues, because we would need two more continuous co-state variables, in addition to keeping track of individid-

[2] Note that this result does not hinge on how exactly agents' outside option is specified. In particular, they may or may not be allowed to store in autarky, and they may or not face additional punishment for defaulting.

[3] We will later show that the first-order condition approach is valid.

ual asset holdings and the co-state variable needed to make the participation constraints recursive.

Instead of adding these state variables, we analyze the social planner's problem ignoring agents' Euler constraints. This provides an important result: if the social planner's Euler constraint is satisfied, which is a necessary condition of optimality, agents no longer have an incentive to save. Therefore, the solution of this simpler problem is identical to the one where agents have a hidden storage opportunity. In other words, hidden storage no longer matters. The intuition behind this result is that the planner has more incentive to save than the agents. First, the planner stores for the agents thereby eliminating their incentives to use the hidden intertemporal technology, and second, storage by the planner makes it easier to satisfy agents' participation constraints in the future. This is due to the fact that, by increasing assets, the planner increases total future consumption available for agents if they stay in the risk sharing arrangement. Since agents are excluded from the benefits of the public asset after default, the higher the amount of accumulated public assets is, the lower agents' incentives to default are. In other words, the planner internalizes the positive effect of aggregate assets on future risk sharing, while the agents do not.

It follows that whenever hidden storage matters, i.e. agents Euler constraints are violated at the constrained-efficient solution of the basic model with storage in autarky, the social planner uses the available storage technology, at least when income inequality is highest. This happens even though there is no aggregate uncertainty and the storage technology is inferior, i.e. the return on storage is lower than the discount rate (or, $\beta(1+r) < 1$). Storage by the planner makes the market more complete in this environment. Note that this is in contrast to the case where private information is the deep friction that limits risk sharing (Cole and Kocherlakota, 2001; Ábrahám and Pavoni, 2008). There, storage by the planner would worsen the information revelation problem, therefore it does not occur in equilibrium.

We further show that whenever hidden storage matters in the basic model with storage in autarky, the stock of public assets is sometimes nonzero and is bounded in the long run. If $\beta(1+r) < 1$, assets remain stochastic in the long run if risk sharing is considerably limited in the sense that participation constraint of each agent binds at more than one income level. We show that in this case there exists an ergodic distribution of assets, that may or may not include zero. Otherwise, assets converge to a strictly positive finite value. Further, given inherited assets, storage will always be higher when cross-sectional consumption inequality is higher. This pattern holds for short-run asset dynamics in general, i.e. also when assets are constant in the long run. The intuition for these results is that the planner's (and agents') saving incentives depend on the consumption distribution. Whenever each agent's participation constraint binds for more than one income level in the long run, these incentives vary over time as the consumption distribution varies over time. This is not true when the participation constraint binds only at the highest income level for each agent, hence assets converge to a steady-state level in that case. If $\beta(1+r) = 1$, it is optimal for the planner to fully complete the market by storage in the long run. That is, perfect risk sharing is self-enforcing in the long run in this case. This is because the trade-off between imperfect insurance and an inefficient intertemporal technology is no longer present.

The introduction of storage has interesting implications for the dynamics of consumption predicted by the model. First, the amnesia property, i.e. that the consumption distribution only depends on the current income of the agent with a binding participation constraint and is independent of the past history of shocks whenever a participation constraint binds (Kocherlakota, 1996), does not hold when assets are stochastic in the long run. The intuition behind this result is that the current consumption distribution depends on both current income and inherited assets, even when a participation constraint binds. In other words, the past history of income realizations affects current consumptions through aggregate assets.

Second, the persistence property of the basic model, i.e. that the consumption allocation does not change for 'small' changes in the income distribution, does not hold either when assets are stochastic in the long run. Even though the sharing rule determining the consumption allocation is the same as in the previous period when neither participation constraint binds, consumption is only constant if net savings are identical in the previous and the current period. This does not happen when assets are stochastic in the long run.s

Data on household income and consumption support neither the amnesia, nor the strong persistence property of the basic model (see Broer (2011) for an extensive analysis). Hence, these differences are steps in the right direction for this framework to explain consumption dynamics.

Third, the Euler constraint, at least in its inequality form, cannot be rejected in micro data from developed economies once labor supply decisions and demographics are appropriately accounted for (Attanasio, 1999). Since in our model the agents' Euler inequalities are satisfied by construction, we bring limited commitment models in line with this observation as well.

In this paper, we also propose an algorithm to solve the model numerically. Using our algorithm, we present some computed examples to illustrate the effects of the availability of storage on risk sharing. We also investigate the consequences of changing the discount factor and the return on storage on the dynamics of assets and consumption. We, however, leave studying the quantitative implications of storage for the dynamics of consumption to future work.

The rest of the paper is structured as follows. Section 2 introduces and characterizes the basic model without access to any intertemporal technology. Section 3 studies whether and under what conditions Euler constraints are binding at the constrained-efficient allocation of the basic model. Section 4 introduces hidden and public storage and derives the main results of the paper. Section 5 presents some computed examples. Section 6 concludes.

## 2 The basic model without storage

We consider an endowment economy with two types of agents, $i = \{1, 2\}$, each of unit measure, who are infinitely lived and risk averse. All agents are ex-ante identical in the sense that they have the same preferences and are endowed with the same exogenous random income process. Agents in the same group are ex-post identical as well, meaning that their income realizations are the same at each time $t$.[4]

Let $u()$ denote the utility function, that is strictly increasing, strictly concave, and continuously differentiable. Assume that the Inada conditions hold.

Let $s_t$ denote the income state realized at time $t$ and $s^t$ the history of income realizations, that is, $s^t = (s_1, s_2, ..., s_t)$. Given $s_t$, agent 1 has income $y(s_t)$, while agent 2 has income equal to $(Y - y(s_t))$, where $Y$ is aggregate income. Note that there is no aggregate uncertainty by assumption. We further assume that income has a discrete support, that is, $s_t \in \{s^1, \ldots, s^j, \ldots, s^N\}$ with $y(s^j) < y(s^{j+1})$, and is independently and identically distributed (i.i.d.), that is, $\Pr(s_t = s^j) = \pi^j$, $\forall t$. The two types of agents framework together with the no aggregate uncertainty assumption imposes some symmetry on both the income realizations and the probabilities. In particular, $y(s^j) = Y - y(s^{N-j+1})$ and $\pi^j = \pi^{N-j+1}$. The i.i.d. assumption can be relaxed as long as weak positive dependence is maintained, i.e. the expected net present value of future lifetime income is weakly increasing in current income.

Suppose that risk sharing is limited by two-sided lack of commitment to risk sharing contracts, i.e. insurance transfers have to be voluntary, or, self-enforcing, as in Thomas and Worrall (1988), Kocherlakota (1996), and others. Each agent may decide at any time and state to default and revert to autarky. This means that only those risk sharing contracts are sustainable which provide a lifetime utility at least as great as autarky after any history

[4] We will refer to agent 1 and agent 2 below. Equivalently, we could say type-1 and type-2 agents, or agent belonging to group 1 and group 2.

of income realizations for each agent. We assume that the punishment for deviation is exclusion from risk sharing arrangements in the future. This is the most severe subgame-perfect punishment in this context. In other words, it is an optimal penal code in the sense of Abreu (1988).

The constrained-efficient risk sharing contract is the solution to the following optimization problem:

$$\max_{c_i(s^t)} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) u\left(c_i\left(s^t\right)\right), \tag{1}$$

where $\lambda_i$ is the (initial) Pareto-weight of agent $i$, $\beta$ is the discount factor, $\Pr(s^t)$ is the probability of history $s^t$ occurring, and $c_i\left(s^t\right)$ is the consumption of agent $i$ when history $s^t$ has occurred; subject to the resource constraints,

$$\sum_{i=1}^{2} c_i\left(s^t\right) \leq Y, \forall s^t, \tag{2}$$

and the participation constraints,

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr\left(s^r \mid s^t\right) u\left(c_i\left(s^r\right)\right) \geq U_i^{au}\left(s_t\right), \forall s^t, \forall i, \tag{3}$$

where $\Pr(s^r \mid s^t)$ is the conditional probability of history $s^r$ occurring given that history $s^t$ occurred up to time $t$, and $U_i^{au}\left(s_t\right)$ is the expected lifetime utility of agent $i$ when in autarky if state $s_t$ has occurred today. In mathematical terms,

$$U_1^{au}\left(s_t\right) = u\left(y(s_t)\right) + \frac{\beta}{1-\beta} \sum_{j=1}^{N} \pi^j u\left(y(s^j)\right) \tag{4}$$

and

$$U_2^{au}\left(s_t\right) = u\left(Y - y(s_t)\right) + \frac{\beta}{1-\beta} \sum_{j=1}^{N} \pi^j u\left(y(s^j)\right).$$

Note that the main qualitative results would remain the same under different punishments as long as the strict monotonicity of the autarky value is maintained. For example, agents could save in autarky (as in Krueger and Perri, 2006), or they might endure additional punishment for defaulting from the community (as in Ligon, Thomas, and Worrall, 2002).

## 2.1 Characterization of the basic model

This model has been well studied in the literature. We briefly derive and describe the main characteristics, in particular those which turn out to be important for our analysis of the interaction between limited commitment and hidden storage. Our characterization is based on the recursive Lagrangian approach of Marcet and Marimon (2011). However, the same results can be obtained using the promised utility approach (see a textbook description in Ljungqvist and Sargent, 2004).

Let $\beta^t \Pr(s^t)\,\mu_i(s^t)$ denote the Lagrange multiplier on the participation constraint, (3), and let $\beta^t \Pr(s^t)\,\gamma(s^t)$ be the Lagrange multiplier on the resource constraint, (2), when history $s^t$ has occurred. Then, the Lagrangian can be written as

$$\begin{aligned}\mathscr{L} = \sum_{t=1}^{\infty}\sum_{s^t}\beta^t \Pr\left(s^t\right)\Bigg\{\sum_{i=1}^{2}\Bigg[&\lambda_i u\left(c_i\left(s^t\right)\right)\\ &+\mu_i\left(s^t\right)\left(\sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\Pr\left(s^r \mid s^t\right)u\left(c_i\left(s^r\right)\right) - U_i^{au}\left(s_t\right)\right)\Bigg]\\ &+\gamma\left(s^t\right)\left(Y - \sum_i c_i\left(s^t\right)\right)\Bigg\}.\end{aligned}$$

Using the ideas of Marcet and Marimon (2011), we can write the Lagrangian in the form

$$\begin{aligned}\mathscr{L} = \sum_{t=1}^{\infty}\sum_{s^t}\beta^t \Pr\left(s^t\right)\Bigg\{&\sum_{i=1}^{2}\left[M_i\left(s^t\right)u\left(c_i\left(s^t\right)\right) - \mu_i\left(s^t\right)U_i^{au}\left(s_t\right)\right]\\ &+\gamma\left(s^t\right)\left(Y - \sum_i c_i\left(s^t\right)\right)\Bigg\},\end{aligned}$$

where $M_i(s^t) = M_i(s^{t-1}) + \mu_i(s^t)$ and $M_i(s^0) = \lambda_i$.

The necessary first-order condition[5] with respect to agent $i$'s consumption when history $s^t$ has occurred is

$$\frac{\partial \mathscr{L}}{\partial c_i(s^t)} = M_i\left(s^t\right)u'\left(c_i\left(s^t\right)\right) - \gamma\left(s^t\right) = 0.$$

Combining such first-order conditions for agent 1 and agent 2, we have

$$x\left(s^t\right) \equiv \frac{M_1\left(s^t\right)}{M_2\left(s^t\right)} = \frac{u'\left(c_2\left(s^t\right)\right)}{u'\left(c_1\left(s^t\right)\right)}. \tag{5}$$

Here $x(s^t)$ is the temporary Pareto weight of agent 1 relative to agent 2.[6] Defining $\upsilon_i(s^t) = \mu_i(s^t)/M_i(s^t)$ and using the definitions of $x(s^t)$ and $M_i(s^t)$, we can obtain the law of motion of $x$ as

$$x\left(s^t\right) = x(s^{t-1})\frac{1-\upsilon_2\left(s^t\right)}{1-\upsilon_1\left(s^t\right)}. \tag{6}$$

[5]Under general conditions, these conditions are also sufficient together with the participation and resource constraints.

[6]To reinforce this interpretation, notice that if no participation constraint binds in history $s^t$ for either agent, i.e. $\mu_1(s^\tau) = \mu_2(s^\tau) = 0$ for all subhistories $s^\tau \subseteq s^t$, then $x(s^t) = \lambda_1/\lambda_2$ is the initial relative Pareto weight of agent $i$.

Notice that given $c_2(s^t) = Y - c_1(s^t)$, the strict concavity of the utility function implies that both consumption levels are uniquely determined by $x(s^t)$ defined in equation (5). In particular, we have that $c_1(s^t)$ is a strictly increasing function of $x(s^t)$. Note also that whenever neither agent's participation constraint binds, i.e. $\upsilon_1(s^t) = \upsilon_2(s^t) = 0$, then the temporary relative Pareto weight, $x$, and consequently the consumption levels, remain constant. This implies that for 'small' changes in the income distribution which do not trigger a binding participation constraint the consumption allocation does not respond. This property is often called an extreme form of *persistence*.

In contrast, when a participation constraint binds, the relative Pareto weight is adjusted. In particular, the relative position of the agent with a binding constraint will improve, i.e. $\upsilon_1(s^t) > 0$ implies $x(s^t) > x(s^{t-1})$, and $\upsilon_2(s^t) > 0$ implies $x(s^t) < x(s^{t-1})$. Moreover, notice that at this point we jump to a new relative Pareto weight which makes him just indifferent between defaulting or not given his current level of income. It is easy to see that this relative Pareto weight only depends on his current income and not the past history. This implies that when the Pareto weights, and hence consumptions, change between periods, the new levels are solely determined by the income level of the agent whose participation constraint is currently binding. The literature often refers to this property as *amnesia*.

The solution of the model is fully characterized by a set of state-dependent intervals on the relative Pareto weight, $x$, which give the possible relative weights in each income state for agent 1. Denote the interval for state $s^j$ by $\left[\underline{x}^j, \overline{x}^j\right]$. $\underline{x}^j$ is defined as the lowest relative Pareto weight such that agent 1, with income $y(s^j)$, is willing to stay in the risk sharing arrangement, and $\overline{x}^j$ is the highest relative Pareto weight such that agent 2, with income $(Y - y(s^j))$, is indifferent between participating and defaulting. In other words, at the lower limit of these intervals agent 1's participation constraint is binding, while at the upper limit agent 2's participation constraint is binding.

Denote by $x_t$ the new relative Pareto weight of agent 1 to be found at time $t$. Suppose that last period the ratio of marginal utilities was $x_{t-1}$, and today the income state is $s^j$. The relative weight of agent 1 today is determined by the following updating rule:

$$x_t = \begin{cases} \overline{x}^j & \text{if} \quad x_{t-1} > \overline{x}^j \\ x_{t-1} & \text{if} \quad x_{t-1} \in \left[\underline{x}^j, \overline{x}^j\right] \\ \underline{x}^j & \text{if} \quad x_{t-1} < \underline{x}^j \end{cases}. \qquad (7)$$

This means that the ratio of marginal utilities is kept constant for any two agents whenever this does not violate the participation constraint of either of them. When the participation constraint binds for agent 1, the relative Pareto weight moves to the lower limit of the optimal interval, just making sure that this agent is indifferent between staying and defaulting.

Similarly, when agent 2's participation constraint binds, the relative Pareto weight moves to the upper limit of the optimal interval. Thereby it is guaranteed that as much risk sharing as possible is achieved while satisfying the participation constraints.

Given that consumption of agent 1 is a strictly increasing function of $x$, exactly the same characterization holds for consumption dynamics. Hence, next we analyze the dynamics of consumption directly. Define the limits of the optimal consumption intervals as

$$\overline{c}^j : \overline{x}^j = \frac{u'\left(Y - \overline{c}^j\right)}{u'\left(\overline{c}^j\right)} \quad \text{and} \quad \underline{c}^j : \underline{x}^j = \frac{u'\left(Y - \underline{c}^j\right)}{u'\left(\underline{c}^j\right)}.$$

It is easy to see that, unless autarky is the only implementable allocation, we have that $\overline{c}^j > \underline{c}^j$, for some $j$.[7] Moreover, given that the value of autarky is increasing in current income (see (4)), it is easy to see that $\overline{c}^j > \overline{c}^{j-1}$ and $\underline{c}^j > \underline{c}^{j-1}$, for all $1 < j \leq N$. Finally, note that symmetry implies that $\overline{c}^j = \underline{c}^{N-j+1}$. The proofs of all these statements can be found in Ljungqvist and Sargent (2004).

Given these results, we will illustrate the dynamics of consumption using a graphical representation. We will focus on scenarios where the long-run equilibrium is characterized by imperfect risk sharing. We do this both because there is overwhelming evidence from several applications (households in a village or in the United States, spouses in a household, countries) about less than perfect risk sharing, and because that case is theoretically not interesting, as it is equivalent to the well-known (unconstrained) efficient allocation of constant individual consumptions over time. Using our notation, perfect risk sharing is obtained in the long run if $\overline{x}^1 \geq \underline{x}^N$, or, equivalently, if $\overline{c}^1 \geq \underline{c}^N$. In other words, if there exists a Pareto weight where neither agent's participation constraint is violated for all income states.

Hence, we assume from now on that $\overline{c}^1 < \underline{c}^N$, i.e. risk sharing is imperfect in the long run. It is not difficult to see that the law of motion described by (7) will imply that risk sharing arrangements subject to limited commitment are characterized by a finite set of consumption values determined by the limits of the optimal consumption intervals. It turns out that, for both the benchmark model and our model with hidden storage, considering two distinct scenarios is enough to describe the general picture: (i) each agent's participation constraint is binding only when his income is highest, and (ii) each agent's participation constraint is binding in more than one state. Further, to describe the constrained-efficient allocations in these two scenarios, it is sufficient to consider three income states, i.e. $N = 3$.

[7] If in autarky agents simply consume their current income every period, then some risk sharing implies $\overline{c}^j > \underline{c}^j$, $\forall j$. If storage is allowed in autarky, then it might be that no transfer is made when cross-sectional income inequality is high, but insurance transfers do occur in states when inequality is low.

Consider an endowment process where each agent gets $y^h$, $y^m$, or $y^l$ units of the consumption good, with $y^h > y^m > y^l$, with probabilities $\pi^h$, $\pi^m$, and $\pi^l$, respectively. Symmetry implies that $y^m = (y^h + y^l)/2$ and $\pi^e \equiv \pi^h = \pi^l = (1-\pi^m)/2$, where the upper index $e$ refers to the most extreme, i.e. most unequal, income distribution. We will refer to a state $s^j$ when agent 1 has income $y^j$.

Given the utility function and the income process, the intervals for different states may overlap or not depending on the discount factor, $\beta$. If $\beta$ is sufficiently large, then perfect risk sharing is self-enforcing by a standard folk theorem (Kimball, 1988). In this case, $\overline{c}^l \geq \underline{c}^h$, and perfect risk sharing is implementable in the long run. If $\beta$ is sufficiently small, there does not exist any non-autarkic allocation that is sustainable with voluntary participation. In this case, each consumption interval collapses to one point, $y^j$, $j = \{l, m, h\}$. For intermediate levels of the discount factor, partial insurance occurs.

If partial insurance occurs, there are two possible scenarios depending in the level of the discount factor. For higher levels of $\beta$, $\overline{c}^m \geq \underline{c}^h > \overline{c}^l \geq \underline{c}^m$. This means that the consumption interval for state $s^m$ overlaps with the intervals associated with both the $s^h$ and the $s^l$ state. This is the case where each agent's participation constraint binds for the highest income level only. Figure 1 presents an example satisfying these conditions.

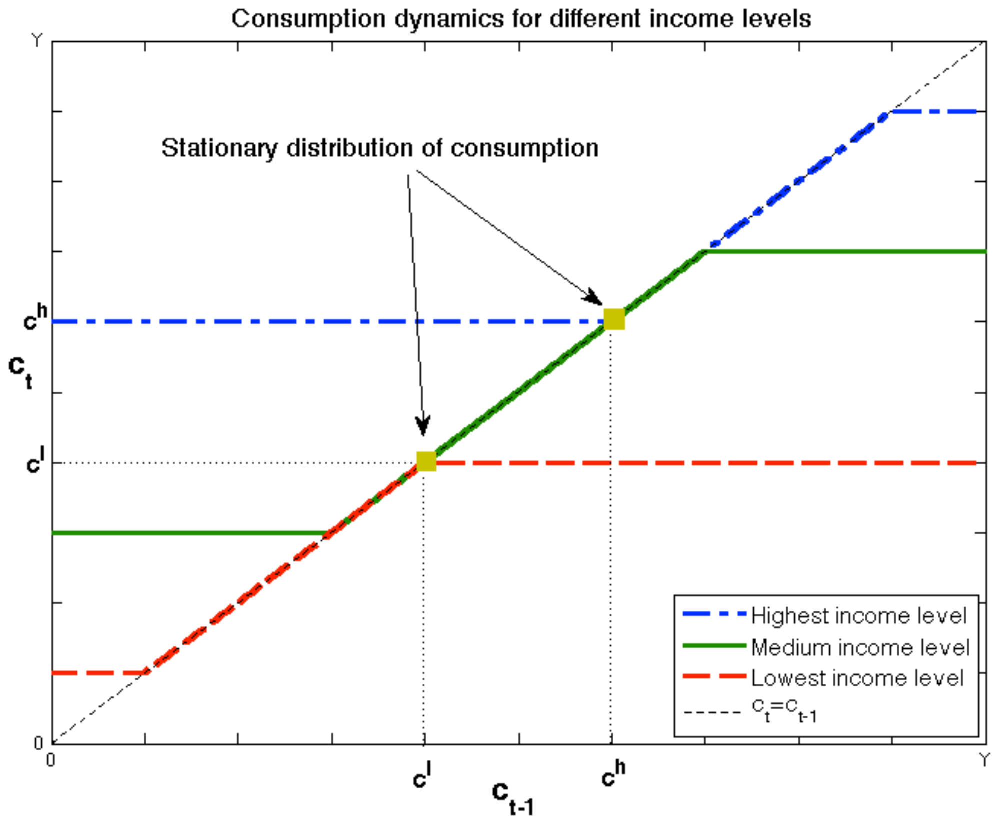
Suppose current consumption of agent 1 is below $\underline{c}^h$. When agent 1 draws a high income realization (which occurs with probability 1 in the long run), his consumption jumps to $\underline{c}^h$. Then it stays at that level until his income jumps to the lowest level. At that period, agent 2's participation constraint binds, because he has high income, and consumption of agent 1 will drop to $\overline{c}^l$. Then we are back to where we started from. This implies that consumption takes only two values, $\underline{c}^h$ and $\overline{c}^l$ in the long run. When consumption changes, it always moves between these two levels, and the past history of income realizations does not matter. This is the way the aforementioned amnesia property presents itself in this case.

When state $s^m$ occurs after state $s^h$ or state $s^l$, the consumption allocation remains unchanged. That is, consumption does not react at all to this 'small' change in income. This is the aforementioned persistence property. Note that consumption also remains unchanged over time if the sequence $(h, m, h)$ or the sequence $(l, m, l)$ takes place.

Another key observation here is that, although individuals face consumption changes over time, the consumption distribution is constant over time. In every period, half of the agents consume $\underline{c}^h$ and the other half consume $\overline{c}^l$. Finally, note that exactly this case occurs for any $N$ if $\overline{c}^2 \geq \underline{c}^N > \overline{c}^1 \geq \underline{c}^{N-1}$.

For lower levels of $\beta$, none of the three intervals overlap, i.e. $\underline{c}^h > \overline{c}^m > \underline{c}^m > \overline{c}^l$. Figure 2

Figure 1: The interval for state $s^m$ overlaps with the intervals for state $s^h$ and state $s^l$

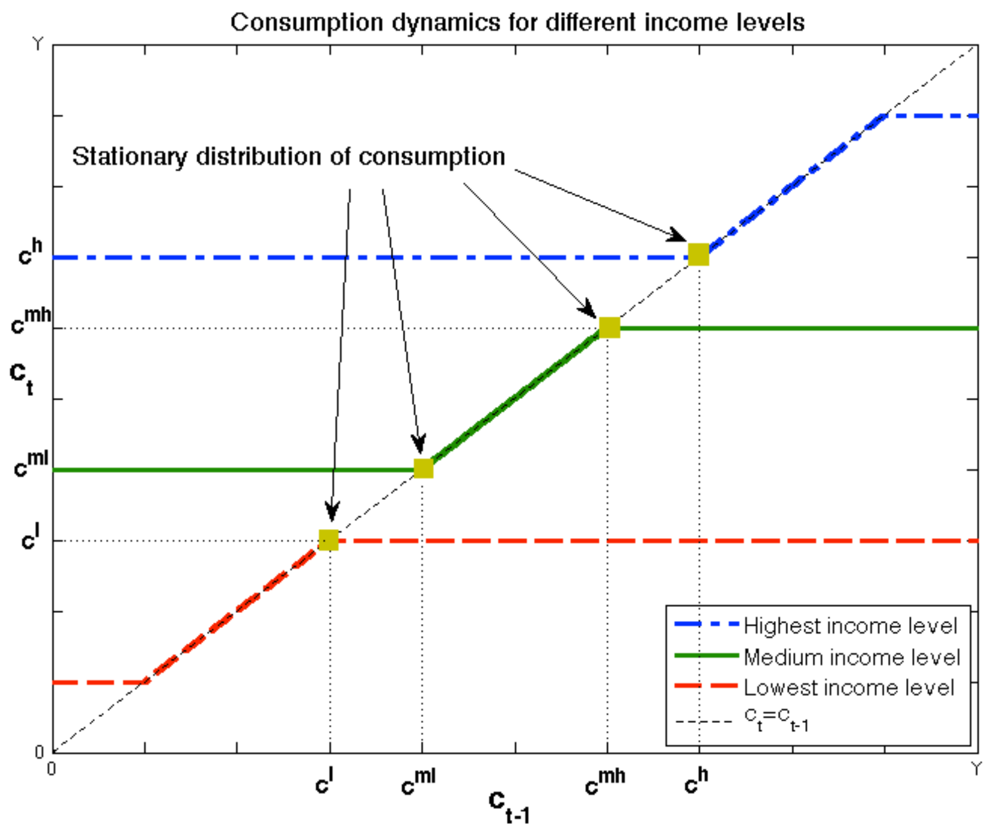



shows an example of this second case. When all three intervals are disjunct, consumption takes four values in the long run. To see this, notice that the participation constraint of agent 1 binds both for medium and high level of income. That is, whenever his income changes his consumption will change as well, and similarly for agent 2.

In this second case, in state $s^m$ the past history determines which agent's participation constraint binds, therefore consumption is Markovian. Current incomes and the identity of the agent with a binding participation constraint fully determine the consumption allocation. The dynamics of consumption exhibit amnesia in this sense here. Further, consumption responds to every income change, hence the persistence property does not manifest itself.

The key observation for later reference is that the consumption distribution changes between $\{\underline{c}^m, \overline{c}^m\}$ and $\{\overline{c}^l, \underline{c}^h\}$. That is, the cross-sectional distribution of consumption is different whenever state $s^m$ occurs from when an unequal income state, $s^h$ or $s^l$, occurs. If there are $N > 3$ income states, the cross-sectional consumption distribution changes over time whenever $\overline{c}^2 < \underline{c}^N$ and $\overline{c}^1 < \underline{c}^{N-1}$. Depending on the number of income states and

Figure 2: The state-dependent intervals are all disjunct




the number of states where a participation constraint binds, the persistence property may appear.

# 3 Does hidden storage matter?

In this section, we study whether and under what conditions agents would save at the constrained-efficient solution of the basic model. We assume that partial insurance occurs at the solution (the interesting case). If Euler constraints are violated, the solution is not robust to deviations when hidden storage is available. We first study the case where the intertemporal technology is *pure storage*, defined as an intertemporal technology that yields zero interest. Then, we consider the other benchmark case where the interest rate is as high as the rate of time preference, i.e. $\beta(1 + r) = 1$, as well as the general case of any return on storage.

## 3.1 Pure storage

When the available intertemporal technology is pure storage, i.e. $r = 0$, we study the two cases we have identified in the previous section: (i) only two consumption levels occur in the long run, i.e. the level of risk sharing is high, and (ii) a participation constraint binds for more than one income level for each agents, or, more than two consumption levels occur in equilibrium, i.e the level of risk sharing is more moderate. We show that they are indeed qualitatively different, as hidden pure storage will matter in the latter but not in the former case.

### 3.1.1 Two consumption levels in the long run

Assume that we are in a situation where, although income can take $N$ values, the optimal allocation of the basic model features only two levels of consumption, $c^h > c^l$. This means that a participation constraint binds only when income inequality is highest. In particular, if the agent's current consumption is $c^h$, then it remains $c^h$ as long as the other agent's income does not reach the highest level. This happens when agent 1's income level is lowest. Hence, the probability of switching to $c^l$ is equal to the probability of the lowest income state, denoted again by $\pi^e$. Therefore, given that an agent's consumption is $c^h$ today, $\pi^e$ is the probability of switching to $c^l$, and $(1 - \pi^e)$ is the probability that his consumption remains $c^h$.

It is easy to see that agents would have an incentive to save/store only if they have high consumption today, because at that point they face a consumption profile which is weakly decreasing. Hence, hidden storage matters at the constrained-optimal allocation if agents' Euler constraints are violated when they consume $c^h$, that is, if we have

$$u'(c^h) < \beta \left[ (1 - \pi^e) \, u'(c^h) + \pi^e u'(c^l) \right] . \tag{8}$$

Given that $c^l = Y - c^h$, it is clear that there exists a level of high consumption, $\hat{c}^h$, where (8) is satisfied with equality:

$$u'(\hat{c}^h) = \beta \left[ (1 - \pi^e) \, u'(\hat{c}^h) + \pi^e u'(Y - \hat{c}^h) \right] . \tag{9}$$

This is the case, because, on the one hand, as $c^h$ is approaching $Y/2$ (and consequently $c^l$), the left hand side of inequality (8) is higher, given $\beta < 1$. On the other hand, as $c^h$ is getting close to $Y$ (and consequently $c^l$ to 0), the right hand side of inequality (8) is higher, because $\lim_{c^l \to 0} u'(c^l) = -\infty$ by the relevant Inada condition. Hence, agents would like to use the pure storage technology in equilibrium if and only if $c^h > \hat{c}^h$. In what follows we will show that this never happens.

In order to do this, we first show that $\hat{c}^h$ is also the level of high consumption which provides the highest life-time utility, given that the agent starts with high consumption.

**Lemma 1.** *$\hat{c}^h$ maximizes welfare for the agent with high consumption today across all possible consumption values if consumption takes only two values.*

*Proof.* The expected lifetime utility of an agent who receives the high consumption, denoted by $\theta^h$, at time $t$ is

$$\begin{aligned} V\left(\theta^h\right) &= u\left(\theta^h\right) + \sum_{\tau=1}^{\infty} \beta^\tau \left[\Pr\left(c_{t+\tau} = \theta^h\right) u\left(c^h\right) + \left(1 - \Pr\left(c_{t+\tau} = \theta^h\right)\right) u\left(Y - \theta^h\right)\right] \\ &= u\left(\theta^h\right) + V\left(\theta^h\right)\left[\beta(1-\pi^e) + \sum_{\tau=2}^{\infty} \beta^\tau (\pi^e)^2 (1-\pi^e)^{\tau-2}\right] \\ &\quad + u\left(Y - \theta^h\right) \sum_{\tau=1}^{\infty} \beta^\tau \pi^e (1-\pi^e)^{\tau-1}. \end{aligned}$$

Then, $V\left(\theta^h\right)$ can be expressed as

$$V\left(\theta^h\right) = \frac{1}{1-\beta} \frac{\left(1 - \beta(1-\pi^e)\right) u\left(\theta^h\right) + \beta\pi^e u\left(Y - \theta^h\right)}{1 - \beta(1 - 2\pi^e)}.$$

The level of consumption, denoted by $\tilde{c}^h$, which maximizes her lifetime utility is given by

$$\tilde{c}^h = \arg\max_{\theta^h} \frac{1}{1-\beta} \frac{\left(1 - \beta(1-\pi^e)\right) u\left(\theta^h\right) + \beta\pi^e u\left(Y - \theta^h\right)}{1 - \beta(1 - 2\pi^e)}.$$

The necessary and sufficient first-order condition for this problem is

$$u'(\tilde{c}^h) = \beta\left[(1-\pi^e)u'(\tilde{c}^h) + \pi^e u'(Y - \tilde{c}^h)\right]. \tag{10}$$

Comparing (9) and (10) the result follows. □

Given this result, it is intuitive that it should never be optimal to implement any consumption level higher than $\hat{c}^h = \tilde{c}^h$. The following proposition proves this statement formally.

**Proposition 1.** *If consumption takes only two values in the long run at the solution of the basic model, then agents have no incentive to use a pure storage technology.*

*Proof.* The proof is by contradiction. Assume that $c^h > \tilde{c}^h$. Then we can construct an alternative allocation which makes both agents better off and does not violate any participation constraint. The alternative contract is given by $\tilde{c}^h$. By construction, this level of consumption makes the agent with high consumption today better off, since $\tilde{c}^h$ maximizes his lifetime utility. The agent with low consumption is also clearly better off, because he enjoys both higher consumption currently, since $c^h > \tilde{c}^h$ implies that $Y - \tilde{c}^h > Y - c^h$, and a reduction of risk in the future, since the allocation $(c^h, Y - c^h)$ is a mean preserving spread of $(\tilde{c}^h, Y - \tilde{c}^h)$. We have increased the lifetime utility of both agents, therefore the participation constraints cannot be violated. This implies that we cannot have a constrained-efficient allocation such that $c^h > \tilde{c}^h$. Then Lemma 1 and equation (9) imply that agents will not have an incentive to use a pure storage technology. □

Proposition 1 means that the optimal allocation will always implement such a low level of consumption variation that agents will have no incentive to use a pure storage technology. It is worth noting that the $N = 2$ case will always fall into this category, because there cannot be more than two levels of consumption in the long-run equilibrium. Second, as we discussed above, more generally this case represents risk sharing situations where a considerable amount of risk sharing is achieved. Proposition 1 shows that this amount of risk sharing is sufficient to eliminate agents' storage incentives, and assets are at their first-best level, zero, in this environment with no aggregate risk.

### 3.1.2 More than two consumption levels in the long run

Let us now consider the case where the participation constraint binds for at least two income levels for each agent. Considering three income states is sufficient to derive our main result. We use the notation which was introduced in Section 2. The key assumption here is that a participation constraint is binding in the intermediate state $s^m$ as well, in addition to the extreme states, $s^h$ and $s^l$, in the long-run equilibrium, see Figure 2.

Consider now the following hypothetical consumption allocation: the high (low) income agent consumes $\theta^h$ $(Y - \theta^h)$ and in state $s^m$ both agents consume $\bar{y} = Y/2$. Let us now define $\hat{c}^h$ as the high consumption level for which the agent's Euler holds with equality if he consumes $\theta^h$, $\bar{y}$, and $Y - \theta^h$ in states $s^h$, $s^m$, and $s^l$, respectively. In mathematical terms, $\hat{c}^h$ is the solution to

$$u'(\hat{c}^h) = \beta\left[\pi^e u'(\hat{c}^h) + \pi^m u'(\bar{y}) + \pi^e u'(Y - \hat{c}^h)\right]. \tag{11}$$

Note that this also means that agents would use the pure storage technology in autarky as long as $y^h > \hat{c}^h$. Similarly, let us now define $\tilde{c}^h$ as the high consumption level that maximizes the lifetime utility of this agent. The first-order condition that characterizes $\tilde{c}^h$ is

$$u'(\tilde{c}^h) = \beta\left[\pi^e u'(\tilde{c}^h) + \pi^e u'(Y - \tilde{c}^h)\right]. \tag{12}$$

The following lemma will be useful to establish whether the Euler constraint may bind at the solution of the basic model with three income states.

**Lemma 2.** $\hat{c}^h < \tilde{c}^h$ *when income takes three values.*

*Proof.* It is enough to verify that the Euler inequality is not satisfied at $\tilde{c}^h$. The right hand side of equation (11) includes an additional positive term, $\pi^m u'(\bar{y})$, compared to equation (12), thus the Euler constraint is violated at $\tilde{c}^h$. □

Note that this result differs from what we have found in the case where consumption takes only two values. In particular, agents would use the intertemporal technology not just for consumption levels above the level that yields the maximum lifetime utility, $\tilde{c}^h$, but also below it as long as $\theta^h > \hat{c}^h$. Using this result for the consumption process $\left(\theta^h, \pi^e; \bar{y}, \pi^m; Y - \theta^h, \pi^e\right)$, we can now state the main result of this subsection.

**Proposition 2.** *Assume that income takes three values, partial insurance occurs, and $y^h$ is sufficiently close to $\tilde{c}^h$. Then, the Euler constraint binds at the constrained-efficient solution of the basic model, even when the intertemporal technology yields no interest.*

*Proof.* Partial insurance occurs if $y^h > \tilde{c}^h$ and $U^{au}(s^h) > u(\bar{y})/(1-\beta)$. This follows from Krueger and Perri (2006), whose results for the two-states case are easy to generalize to three states. For $y^h > \tilde{c}^h$ close to $\tilde{c}^h$, $c^h < \tilde{c}^h$ is in a small neighborhood of $\tilde{c}^h$, and consumption in state $s^m$ is in a small neighborhood of $\bar{y}$. The result follows from Lemma 2 by continuity. □

It is natural that, if for some high consumption level agents would deviate from the constrained-efficient allocation by using the pure storage technology, then this happens when the solution is close to autarky, i.e. when state-contingent insurance transfers are small, and agents bear a lot of consumption risk.

This result concerning pure storage is interesting because it is hard to exclude that such a technology is available to agents in several economic contexts that the risk sharing with limited commitment model has been applied to. For example, think of households in poor villages or members of a household who can hide grain or cash.

## 3.2 Storage with any return

In this section we first consider the benchmark case where agents have access to an efficient intertemporal technology, i.e. storage earns a return $r$ such that $\beta(1+r) = 1$. Afterwards, we study the general case. As above, we only examine whether agents would use the available hidden intertemporal technology at the constrained-efficient solution of the basic model. We do not make any assumption about the number of income states, except that income may take a finite number of values, and the support of the income distribution is bounded.

**Lemma 3.** *Suppose that partial insurance occurs and the hidden storage technology yields a return $r$ such that $\beta(1+r) = 1$. Then the Euler constraint is violated at the constrained-efficient allocation when an agent receives the highest possible income, $y^N$.*

*Proof.* If partial insurance occurs, as opposed to full insurance, then it must be that there exists some state $s^{\tilde{j}}$ where the agent consumes $c^{\tilde{j}} < c^N$. Then,

$$u'(c^N) < \sum_{s^j} \Pr(s^j) u'(c^j),$$

that is, the Euler constraint is violated. □

**Proposition 3.** *There exists $\tilde{r} < 1/\beta - 1$ such that for all $r > \tilde{r}$, agents' Euler constraints are violated at the constrained-efficient allocation of the basic model.*

*Proof.* $\tilde{r}$ is defined as the solution to

$$u'(c^N) = \beta(1+\tilde{r}) \sum_{s^j} \Pr(s^j) u'(c^j). \tag{13}$$

For $\tilde{r}$ close to $-1$, the right hand side is close to zero. By Lemma 3, the right hand side is greater than the left hand side if $\tilde{r} = \frac{1}{\beta} - 1$. It is obvious that the right hand side is increasing in $\tilde{r}$. Therefore, there is a unique $\tilde{r}$ solving equation (13), and agents' Euler constraint is violated for higher values of $r$. □

Having established that Euler constraints bind for a sufficiently high $r$, where 'high' can mean $r = 0$, in the next section we turn to the general problem to find the constrained-efficient risk sharing contract satisfying both participation and Euler constraints.

# 4 The model with storage

In this section, we provide the formulation and analytical characterization of our model with limited commitment and hidden storage. We add agents' Euler constraints to the problem given by the objective function (1) and the constraints (2) and (3). We also modify the resource constraint to allow the social planner to use the same intertemporal technology as the agents. We maintain the assumption of no aggregate uncertainty to exclude the precautionary motive for saving at the aggregate level, and to thus isolate the saving incentive in limited commitment models due to endogenously incomplete markets.

The social planner's problem is

$$
\begin{aligned}
& \max_{\{c_i(s^t),B(s^t)\}} \sum_{i=1}^{2} \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr\left(s^t\right) u\left(c_i\left(s^t\right)\right) && (14)\\
& \text{s.t.} \quad \sum_{i=1}^{2} c_i\left(s^t\right) \leq \sum_{i=1}^{2} y_i\left(s_t\right) + (1+r)B\left(s^{t-1}\right) - B\left(s^t\right), \forall s^t, && (15)\\
(P1) \quad & \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr\left(s^r \mid s^t\right) u\left(c_i\left(s^r\right)\right) \geq \tilde{U}_i^{au}\left(s_t\right), \forall s^t, \forall i, && (16)\\
& u'\left(c_i\left(s^t\right)\right) \geq \beta(1+r) \sum_{s^{t+1}} \Pr\left(s^{t+1} \mid s^t\right) u'\left(c_i\left(s^{t+1}\right)\right), \forall s^t, \forall i, && (17)\\
& B\left(s^t\right) \geq 0, \forall s^t. && (18)
\end{aligned}
$$

The social planner chooses the consumption of each agent given any possible history of income states, $c_i\left(s^t\right)$, $\forall i$, and current gross storage given any possible history of income states, $B\left(s^t\right)$, and maximizes a weighted sum of agents' lifetime utilities. The first constraint, (15), is the resource constraint, where $B\left(s^{t-1}\right)$ denotes assets inherited from the previous period. The next constraint, (16), is the participation constraint, where $\tilde{U}_i^{au}\left(s_t\right)$ is the value function of autarky when storage is allowed. Equation (17) is agents' Euler constraint. Finally, (18) is the planner's borrowing constraint.

A few remarks are in order about this structure before we turn to the characterization of constrained-efficient allocations. First, agents can store in autarky, but they lose access to the benefits of the public asset.[8] This implies that $\tilde{U}_i^{au}\left(s^j\right) = V_i^{au}\left(s^j, 0\right)$, where $V_i^{au}\left(s^j, b\right)$ is defined as

$$
V_i^{au}\left(s^j, b\right) = \max_{b'} \left\{ u(y_i(s^j) + (1+r)b - b') + \beta \sum_{k=1}^{N} \pi^k V_i^{au}\left(s^k, b'\right) \right\}, \tag{19}
$$

where $b$ denotes private savings. Since $V_i^{au}\left(s^j, 0\right)$ is increasing (decreasing) in $j$ for agent 1 (2), it is obvious that if we replace for the autarky value in the basic model introduced in Section 2 with the one defined here, the same characterization holds, given that a solution exists.

Second, we allow for public storage exactly in order to assure that a solution exists. In other words, this assumption makes sure that the feasible set is nonempty. If neither the social planner nor the agents can store in the optimal contract, but agents can and would store in autarky, then no feasible solution exists for sufficiently low $\beta$s, i.e. when the solution

[8] This is the same assumption as in Krueger and Perri (2006), where agents lose access to the benefits of a tree after defaulting.

of the basic model is 'close' to autarky. Note, however, that since the return on public and private storage is the same, public storage can replicate any allocation which agents can achieve using private storage. Further, equation (19) implicitly assumes that public storage does not affect the value of autarky. As mentioned above, when agents default they are not only excluded from future risk sharing but also from the benefits of the public asset. This implies that public storage has an additional benefit: it makes default a relatively less attractive option. Last but not least, public storage is relevant for applications as well. One example is the existing community grain storage facilities in low-income villages. Another example is the European Financial Stability Facility and the European Stability Mechanism (to be launched), which can facilitate risk sharing across countries within the euro area. If a country left the euro area, it would lose access to these resources.

Third, we use a version of the first-order condition approach (FOCA) here. That is, we only check single deviations. In particular, we check whether the agent is better off staying in the risk arrangement or defaulting given that he does not store (condition (16)), and we check whether he is happy with not storing given that he does not default (condition (17)). It is not obvious whether these conditions are sufficient.[9] In principle, it is possible that the agent stores in the current period to increase his value of autarky in future periods, and defaults in a later period. Our constraints do not take such deviations into account. For now this is an assumption. Given this assumption, we will characterize the solution. Afterwards, we will show that agents indeed have no incentive to use these more complex deviations in Section 4.2.

Fourth, both the participation constraints (16) and the Euler constraints (17) involve future decision variables. Given these two types of forward-looking constraints, a recursive formulation using either the promised utilities approach (Abreu, Pearce, and Stacchetti, 1990) or the Lagrange multipliers approach (Marcet and Marimon, 2011) is difficult. Euler constraints have been dealt with in models with moral hazard and hidden storage using the agent's marginal utility as a co-state variable, see Werning (2001) and Ábrahám and Pavoni (2008). In our environment, this could raise serious tractability issues, since we would need two more continuous co-state variables, in addition to the state variables to keep track of individual asset holdings.

In this paper, we follow a different approach that avoids these complications. In particular, we solve the problem ignoring agents' Euler constraints first. Then we verify that the solution of the simplified problem satisfies those Euler constraints. That is, instead of Problem *P1*,

[9] In fact, Kocherlakota (2004) shows that in an economy with private information and hidden storage the first-order condition approach can be invalid.

we solve the following simpler problem:

$$
(P2)\qquad
\begin{aligned}
\max_{\{c_i(s^t),B(s^t)\}} & \sum_{i=1}^{2}\lambda_i\sum_{t=1}^{\infty}\sum_{s^t}\beta^t \Pr\left(s^t\right)u\left(c_i\left(s^t\right)\right)\\
\text{s.t.}\quad & \sum_{i=1}^{2}c_i\left(s^t\right)\le\sum_{i=1}^{2}y_i\left(s_t\right)+(1+r)B\left(s^{t-1}\right)-B\left(s^t\right),\forall s^t,\\
& \sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\Pr\left(s^r\mid s^t\right)u\left(c_i\left(s^r\right)\right)\ge\tilde{U}_i^{au}\left(s_t\right),\forall s^t,\forall i,\\
& B\left(s^t\right)\ge 0,\forall s^t.
\end{aligned}
$$

Next, we write Problem *P2* in a recursive form. Let $\beta^t \Pr(s^t)\,\mu_i(s^t)$ denote the Lagrange multiplier on the participation constraint, (16), and let $\beta^t \Pr(s^t)\,\gamma(s^t)$ be the Lagrange multiplier on the resource constraint, (15), when history $s^t$ has occurred, as in Section 2. Then, the Lagrangian is

$$
\begin{aligned}
\mathscr{L}=&\sum_{t=1}^{\infty}\sum_{s^t}\beta^t\Pr\left(s^t\right)\left\{\sum_{i=1}^{2}\left[\lambda_i u\left(c_i\left(s^t\right)\right)\right.\right.\\
&\left.+\mu_i\left(s^t\right)\left(\sum_{r=t}^{\infty}\sum_{s^r}\beta^{r-t}\Pr\left(s^r\mid s^t\right)u\left(c_i\left(s^r\right)\right)-\tilde{U}_i^{au}\left(s_t\right)\right)\right]\\
&\left.+\gamma\left(s^t\right)\left(\sum_{i=1}^{2}\left(y_i\left(s_t\right)-c_i\left(s^t\right)\right)+(1+r)B\left(s^{t-1}\right)-B\left(s^t\right)\right)\right\}
\end{aligned}
$$

Similarly as for the basic model, we can rewrite the Lagrangian as

$$
\begin{aligned}
\mathscr{L}=&\sum_{t=1}^{\infty}\sum_{s^t}\beta^t\Pr\left(s^t\right)\left\{\sum_{i=1}^{2}\left[M_i\left(s^t\right)u\left(c_i\left(s^t\right)\right)-\mu_i\left(s^t\right)\tilde{U}_i^{au}\left(s_t\right)\right]\right.\\
&\left.+\gamma\left(s^t\right)\left(\sum_{i=1}^{2}\left(y_i\left(s_t\right)-c_i\left(s^t\right)\right)+(1+r)B\left(s^{t-1}\right)-B\left(s^t\right)\right)\right\}.
\end{aligned}
$$

where $M_i(s^t)=M_i(s^{t-1})+\mu_i(s^t)$ and $M_i(s^0)=\lambda_i$, as before. This Lagrangian is recursive with current income realizations, inherited assets, and the relative Pareto weight from last period serving as state variables, and the current relative Pareto weight and storage serving as controls.

## 4.1 Characterization

The first-order condition with respect to agent $i$'s consumption when history $s^t$ has occurred is

$$
\frac{\partial\mathscr{L}}{\partial c_i\left(s^t\right)}=M_i\left(s^t\right)u'\left(c_i\left(s^t\right)\right)-\gamma\left(s^t\right)=0. \tag{20}
$$

Combining such first-order conditions for agent 1 and agent 2, and using the definitions of $M_i(s^t)$ and of $x(s^t)$, we have

$$x\left(s^t\right) = \frac{M_1\left(s^t\right)}{M_2\left(s^t\right)} = \frac{u'\left(c_2\left(s^t\right)\right)}{u'\left(c_1\left(s^t\right)\right)}. \tag{21}$$

Remember that $\upsilon_i(s^t) = \mu_i(s^t)/M_i(s^t)$. Then, the law of motion of $x$ is

$$x\left(s^t\right) = x(s^{t-1})\frac{1-\upsilon_2\left(s^t\right)}{1-\upsilon_1\left(s^t\right)}, \tag{22}$$

as in the basic model.

The planner's Euler constraint, i.e. the optimality condition for $B(s^t)$ is

$$\gamma\left(s^t\right) \geq \beta(1+r)\sum_{s^{t+1}}\beta^t \Pr\left(s^{t+1}|s^t\right)\gamma\left(s^{t+1}\right), \tag{23}$$

which, using (20), can also be written as

$$M_i\left(s^t\right)u'\left(c_i\left(s^t\right)\right) \geq \beta(1+r)\sum_{s^{t+1}}\Pr\left(s^{t+1} \mid s^t\right)M_i\left(s^{t+1}\right)u'\left(c_i\left(s^{t+1}\right)\right).$$

Then, using (21) and (22), the planner's Euler becomes

$$u'\left(c_i\left(s^t\right)\right) \geq \beta(1+r)\sum_{s^{t+1}}\Pr\left(s^{t+1}|s^t\right)\frac{u'\left(c_i\left(s^{t+1}\right)\right)}{1-\upsilon_i\left(s^{t+1}\right)}, \tag{24}$$

where $0 \leq \upsilon_i(s^{t+1}) \leq 1$. Given the definition of $\upsilon_i(s^{t+1})$ and equation (22), it is easy to see that (23) represents exactly the same mathematical relationship for both agents. Comparing the planner's Euler, (24), to the standard Euler constraints for the agents gives the following result:

**Proposition 4.** *When the planner's Euler is satisfied, the agents' Eulers are satisfied as well. Therefore, the solution of the model with hidden storage, P1, corresponds to the solution of the simplified problem, P2.*

*Proof.* The planner's Euler is (24), while the agents' Euler is

$$u'\left(c_i\left(s^t\right)\right) \geq \beta(1+r)\sum_{s^{t+1}}\Pr\left(s^{t+1}|s^t\right)u'\left(c_i\left(s^{t+1}\right)\right). \tag{25}$$

The right hand side of (24) is bigger than the right hand side of (25), for $i = \{1, 2\}$, since $0 \leq \upsilon_i(s^{t+1}) \leq 1$, $\forall s^{t+1}$. Therefore, (24) implies (25). □

**Corollary 1.** *The planner stores in equilibrium whenever an agent's Euler constraint is violated at the constrained-efficient allocation of the basic model with storage in autarky.*

Note that the public storage technology is used even though there is no aggregate uncertainty and the technology is inferior ($r < 1/\beta - 1$). Intuitively, this is optimal for two interrelated reasons. First, given that an agent would have an incentive to store, the planner can eliminate it by public storage. Note that when an agent has the highest level of consumption, the planner's Euler is identical to the agent's Euler. This is easy to see comparing (24) and (25). Hence, in this case, the only reason for public asset accumulation is to store for the high-income agent. This also implies that this will be the only reason for using the available intertemporal technology in the case with a high level of risk sharing, i.e. where only two consumption levels occur in the long run.

Second, storage by the planner makes it easier to satisfy the participation constraints next period, thus it improves risk sharing in the future. This is the case whenever a participation constraint binds in other states as well, not just when income inequality is highest. Comparing (24) and (25) again, it is obvious that the planner has more incentive to save than the agents in the other states. In particular, the presence of $1/\left(1-\upsilon_i\left(s^{t+1}\right)\right\rangle 1$ in the planner's Euler exactly indicates how increasing assets helps the planner to relax future participation constraints, and thereby improve future risk sharing.

Given Proposition 4, we can focus on solving problem *P2*, which will also give us the solution of problem *P1*. Next, we introduce some useful notation and show more precisely the recursive formulation of *P2*, which we will use to solve the model numerically. Let $y$ denote the current income of agent 1, and $V()$ denote the value function. The following system is recursive using $X = (y, B, x)$ as state variables:

$$x'(X) = \frac{u'\left(Y + (1+r)B - B'(X) - c_1(X)\right)}{u'\left(c_1(X)\right)} \tag{26}$$

$$x'(X) = x\frac{1-\upsilon_2(X)}{1-\upsilon_1(X)} \tag{27}$$

$$u'\left(c_1(X)\right) \geq \beta(1+r)\sum_{y'} \Pr\left(y'\right)\frac{u'\left(c_1(X')\right)}{1-\upsilon_1(X')} \tag{28}$$

$$u\left(c_1(X)\right) + \beta\sum_{y'} \Pr\left(y'\right) V\left(X'\right) \geq \tilde{U}^{au}\left(y\right) \tag{29}$$

$$u\left(Y + (1+r)B - B'(X) - c_1(X)\right) + \beta\sum_{y'} \Pr\left(y'\right) V\left(Y - y', B', 1/x'\right) \geq \tilde{U}^{au}\left(Y - y\right) \tag{30}$$

$$B'(X) \geq 0. \tag{31}$$

The first equation, (26), says that the ratio of marginal utilities between the two agents has to be equal to the current relative Pareto weight, and where we have used the resource constraint to substitute for $c_2(X)$. Equation (27) is the law of motion of the co-state variable, $x$. Equation (28) is the social planner's Euler constraint, which we have derived above. Equations (29) and (30) are the participation constraints of agent 1 and agent 2, respectively, where $\tilde{U}^{au}(y) = \tilde{U}_1^{au}(s^j)$ and $\tilde{U}^{au}(Y - y) = \tilde{U}_2^{au}(s^j)$, with $y = y^j = y_1(s^j)$. Finally, equation (31) is the planner's borrowing constraint.

Given the recursive formulation above, and noting that the outside options $\tilde{U}^{au}(y)$ and $\tilde{U}^{au}(Y - y)$ are monotone in $y$ and take a finite set of values, the solution can be characterized by a set of state-dependent intervals, as in the basic model. This is straightforward in the (for now hypothetical) case of constant and positive level of public savings, $B^* > 0$. In this case, we can repeat the same analysis as in Section 2 with only two differences: (i) the autarky value is $\tilde{U}_i^{au}()$ instead of $U_i^{au}()$, and (ii) aggregate consumption is given by $Y + rB^*$ instead of $Y$. This implies that the same updating rule applies as in the basic model, see (7).

In the case where public assets are not constant, the consumption intervals naturally become a function of the changing level of available aggregate resources, $Y + (1 + r)B$. Nevertheless, optimal state-dependent intervals on the relative Pareto weight still characterize the solution, but they depend on $B$ as well, not just on current income realizations. To see this, note that we can express the value function in terms of the end of the period relative Pareto weight and the inherited asset level only. This is without loss of generality because of equation (27). It follows that the following conditions define the lower and upper bound of these intervals:

$$V(y^j, \underline{x}^j(B), B) = \tilde{U}^{au}\left(y^j\right) \quad \text{and} \quad V\left(Y - y^j, \frac{1}{\overline{x}^j(B)}, B\right) = \tilde{U}^{au}\left(Y - y^j\right).$$

Hence, given the inherited Pareto weight, $x_{t-1}$, and accumulated assets, $B$, the updating rule (7) needs to be modified as follows:

$$x_t = \begin{cases} \overline{x}^j(B) & \text{if} \quad x_{t-1} > \overline{x}^j(B) \\ x_{t-1} & \text{if} \quad x_{t-1} \in \left[\underline{x}^j(B), \overline{x}^j(B)\right] \\ \underline{x}^j(B) & \text{if} \quad x_{t-1} < \underline{x}^j(B) \end{cases}. \tag{32}$$

Since this updating rule is practically the same as the one for the basic model (compare (7) and (32)), we can determine the level of risk sharing in this model as well, given $B$, by checking how much these intervals overlap. Despite the fact that the transition rules are remarkably similar to the benchmark case, there is a very important difference: as the aggregate storage level changes, these intervals also change over time, which implies that we do not necessary stay at the limits of the optimal intervals even in the long run. To see this, notice that whenever $B' > B$, the relative attractiveness of autarky is reduced, hence

$\underline{x}^j(B) > \underline{x}^j(B')$ and $\overline{x}^j(B) < \overline{x}^j(B')$. This implies that if the income state is the same today as yesterday, we have that the $\underline{x}^j(B') < x_{t-1} < \overline{x}^j(B')$, thus $x_t = x_{t-1}$, and $x_t$ may not be at the bound of any state-dependent interval given that inherited assets are $B'$.

### 4.1.1 The dynamics of aggregate assets

Next, we further characterize the evolution of aggregate assets. Given the law of motion of the relative Pareto weight defined above, this will not only provide us with the dynamics of aggregate consumption, but also of individual consumptions. Recall that the only reason for which the planner saves in the high risk sharing (two consumption levels only) case is to prevent the high consumption agent from using private storage. In contrast, when risk sharing is more limited, the participation constraint binds at several levels of income. The planner's saving incentives vary across these income levels both because agents' private saving incentives vary and because at lower levels of income she has more incentives to save than the agents in order to relax future participation constraints. The next proposition will show that this difference is not so important for the short-run dynamics of assets but has very significant implications for the long-run dynamics.

Before the main proposition, we provide some useful properties of the policy function for storage. First, notice that (24) implies that the optimal choice of assets is solely determined by the end of period Pareto weight and inherited assets, hence we can write $B'(B, x')$. The following lemma makes this statement more precise:

**Lemma 4.** $B'(s^j, B, x) = B'(B, x')$. *That is, for determining aggregate storage, the current relative Pareto weight* $x'$ *is a sufficient statistic for the current income state,* $s^j$*, and last period's relative Pareto weight,* $x$*.*

*Proof.* Once we know $x'$, equations (26) and (28), which do not depend on $x$, give $c_1$ and $B'$. □

The next lemma provides a key property of the aggregate storage decision rule.

**Lemma 5.** $B'(B, x')$ *is strictly increasing in* $x'$ *for* $x' \geq 1$ *and* $B'(B, x') > 0$*. That is, the higher cross-sectional consumption inequality is, the higher public asset accumulation is.*

*Proof.* A higher $x' \geq 1$ increases public storage for two complimentary reasons. First, higher consumption inequality, which is uniquely determined by the current relative Pareto weight $x'$, increases the private saving incentive of the agent with high current consumption. To make sure that the high-consumption agent does not deviate by hidden storage, the planner has to save more when $x' > 1$ is higher. Second, remember that the planner's additional saving incentives come from the fact that she aims to improve risk sharing (reduce consumption inequality) in the future. A higher $x' > 1$ implies higher consumption inequality tomorrow. To see this, note that three things can happen tomorrow with respect to the pattern of binding participation constraints: (i) no participation constraint binds tomorrow, thus consumption inequality remains the same as today, (ii) agent 1's participation constraint is binding, thus consumption inequality either remains higher for a higher $x' > 1$ or no longer depends on $x'$ when comparing a higher and a lower $x'$, and (iii) agent 2 participation constraint is binding, and either $x'' > 1$ and the same cases are possible as for (ii), or $x'' < 1$ in which case consumption inequality tomorrow does not depend on $x'$. Therefore, to reduce inequality tomorrow, the planner has higher incentives to save when $x' > 1$ is higher as well. □

We are now ready to characterize the long-run behavior of aggregate assets.

**Proposition 5.**

*(i) $B$ converges almost surely to a strictly positive constant in the long run whenever the agents' Euler constraints are violated at $B' = B = 0$ for some income distribution, $r < 1/\beta - 1$, and each agent's participation constraint binds only when his income is highest in the long run.*

*(ii) $B$ is stochastic and bounded in the long run whenever the agents' Euler constraints are violated at $B' = B = 0$ for some income distribution, $r < 1/\beta - 1$, and each agent's participation constraint binds in more than one income state in the long run.*

*(iii) $B$ converges almost surely to a strictly positive constant $\widehat{B}$ in the long run such that $\forall B \geq \widehat{B}$ perfect-risk sharing is self-enforcing, whenever the agents' Euler constraints are violated at $B' = B = 0$ for some income distribution and $r = 1/\beta - 1$. If the initial level of assets is above $\widehat{B}$, then aggregate assets stay constant at that level.*

*Proof.* First, from Corollary 1 we know that the planner's Euler constraint, (24), is violated under the assumption that the agents' Euler constraints are violated at $B' = B = 0$ for some $x' = \hat{x}$, where $\hat{x}$ is included in the long run ergodic set of relative Pareto weights. Therefore, $B'(0, \hat{x}) > 0$. It is easy to see that there exists a high level of inherited assets, denoted $\widehat{B}$, such that perfect intertemporal insurance is at least temporarily enforceable, that is, $\overline{x}^1\left(\widehat{B}\right) \geq \underline{x}^N\left(\widehat{B}\right)$. Therefore, $B'(B, x') < B$ for all $B \geq \widehat{B}$ and $\overline{x}^1(B) \geq x' \geq \underline{x}^N(B)$, i.e. assets have to optimally decrease, because $r < 1/\beta - 1$ by assumption. This implies that assets are bounded above in the long run.

For part (i), we first show that there exists a unique constant level of assets, $B^*$, such that all optimality conditions are satisfied. Afterwards, we will show that assets converge almost surely to $B^*$ starting from any initial level, $B_0$.

First, recall that if aggregate assets are constant, the optimal intervals for the relative Pareto weight are time invariant. Given that each agent's participation constraint binds only for the highest income level in the long run, the optimality condition (26) and $\underline{x}^N(B^*)$ uniquely determine $c^h(B^*)$, the time-invariant 'high' consumption level. Then, using the planner's Euler, we can determine the unique level of $B^*$ such that all optimality conditions are satisfied. The planner's Euler is

$$u'\left(c^h\left(B^*\right)\right) = \beta(1+r)\left[\left(1-\pi^e\right)u'\left(c^h\left(B^*\right)\right) + \pi^e u'\left(c^l\left(B^*\right)\right)\right].$$

Dividing both sides by $u'\left(c^h\left(B^*\right)\right)$, we obtain

$$\begin{aligned} 1 &= \beta(1+r)\left[\left(1-\pi^e\right) + \pi^e \frac{u'\left(c^l\left(B^*\right)\right)}{u'\left(c^h\left(B^*\right)\right)}\right] \\ &= \beta(1+r)\left[\left(1-\pi^e\right) + \pi^e \underline{x}^N\left(B^*\right)\right], \end{aligned} \tag{33}$$

where we have used (26). Note that $\underline{x}^N\left(B^*\right)$ is monotone and continuous in $B^*$. Further, at $B^* = 0$ the right hand side of equation (33) is larger than 1 by assumption, and at $B^* = \widehat{B}$ the right hand side of (33) is smaller than 1, because $\underline{x}^N(\widehat{B}) = 1$ and $B^* < \widehat{B}$. Therefore, we know that there exists a unique $B^*$ where the planner's Euler is exactly satisfied by setting $B' = B = B^*$.

Next, we show that assets converge almost surely to $B^*$ starting from any initial asset level, $B_0$. We already know that $B'(B_0, x') < B_0$ for the ergodic range of $x'$ when $B_0 > \widehat{B}$, i.e. when perfect risk sharing is (temporarily) self-enforcing, and $B'(0, x') > 0$ for some $x'$ in the ergodic range of $x'$, since we have assumed that agents' Euler constraints are binding in the basic model. Consider $B^* < B_0 < \widehat{B}$ first, and assume that state $N$ occurs, and agent 1's participation constraint is binding. This is without loss of generality, because this occurs with probability 1 in the long run, and the problem is symmetric across the two agents. We know that the right hand side of (33) is smaller than 1, because $\underline{x}^N\left(B_0\right) < \underline{x}^N\left(B^*\right)$. Therefore, marginal utility tomorrow has to increase relative to marginal utility today to satisfy the planner's Euler, therefore $B'(B_0) < B_0$. What happens next period? The participation constraint will bind again even if the same state occurs.[10] This is because $B'(B_0) < B_0$ implies $\underline{x}^N\left(B'(B_0)\right) > \underline{x}^N\left(B_0\right)$. Then assets will decrease again. What if some $s^j$ with

[10]Note that this never happens in the basic model.

$2 \leq j \leq N-1$ occurs? We know that the participation constraints in these states are not binding for any $B \geq B^*$, because they are not binding for $B^*$. This means that now $x' = x = \underline{x}^N(B_0) < \underline{x}^N(B'(B_0))$. Then, by Lemma 5, storage will be lower than when the participation constraint is binding. Note that if states $s^2, ..., S^{N-1}$ occur repeatedly, assets will converge to a level below $B^*$. Then we are in the case where $B_0 < B^*$, that we study next.

Consider $0 \leq B_0 < B^*$ now, and suppose again that state $N$ occurs, and agent 1's participation constraint is binding. We know that $\underline{x}^N(B_0) > \underline{x}^N(B^*)$ in this case. Using (33) again, it follows that $B'(B_0) > B_0$. Now, if the same state occurs tomorrow (in fact, any $s^j$ with $j \geq 2$), then the participation constraint will be slack. This means that now $x' = x = \underline{x}^N(B_0) > \underline{x}^N(B'(B_0))$. Then, by Lemma 5, storage will be higher than when the participation constraint is binding. This also implies that if state $s^1$ does not occur for many periods, assets converge to a level above $B^*$. Then once $s^1$ occurs, which happens with probability 1 in the long run, we are back to the case $B_0 > B^*$, and assets start decreasing.[11]

To see part (ii), consider the case where in the long run there is a third state in which a participation constraint binds. In this case, each agent's consumption takes at least four different values in the long run. These have to satisfy an additional participation constraint, an additional resource constraint, and an additional Euler, which is generically impossible for constant $B$.

To see part (iii), note that when $\beta(1+r) = 1$, the only way to satisfy agents' Euler constraints in all states is to provide them with a perfectly smooth consumption stream over time. Further, as long as a participation constraint binds given $B$, the planner has an incentive to save more, because she does not face a trade-off between improving risk sharing and using an inefficient intertemporal technology. If the $B_0$ is higher than the minimum level necessary to satisfy all participation constraints, $B' = B$ clearly satisfies both the agents' and the planner's Euler constraints, thus assets will remain constant at $B_0 \geq \widehat{B}$. □

Proposition 5 says that aggregate assets will be constant in the long run if the cross-sectional consumption distribution does not change over time. This is what happens not just when consumption takes only one value, but also when it takes two values in the long run. In the latter case half the agents consume $c^h$ and the other half consume $c^l$ in each period, only the identity of the agent with $c^h$ changes over time. The social planner trades off two

[11] Participation constraints in more states may be binding when $B$ is low, even if they only bind in states $s^1$ and $s^N$ for $B^*$. We know that assets will increase in the two most unequal states when $B < B^*$, therefore with probability 1 assets will reach a level where the participation constraints of the other states are no longer binding.

effects of increasing aggregate storage: it is costly because $\beta(1+r) < 1$, but it is beneficial because it reduces consumption dispersion in the future and discourages private storage. The steady-state level of assets just balances these two opposing forces.

When participation constraints bind in more states, and consumption has to take more than two values, the cross-sectional consumption distribution will change over time. Thus, the relative strength of these forces also changes over time, implying that assets remain stochastic in the long run. The following proposition shows how exactly public storage varies with the income and consumption distribution.

**Proposition 6.** *$B'(s^j, B, x) \geq B'(s^k, B, x)$, $\forall (B, x)$, where $j \geq N/2+1$, $k \geq N/2$, and $j > k$. The inequality is strict, i.e. $B'(s^j, B, x) > B'(s^k, B, x)$, if the optimal intervals for states $s^j$ and $s^k$ do not overlap given $B$. That is, given inherited assets and the relative Pareto weight last period, aggregate storage is greater when consumption inequality is higher.*

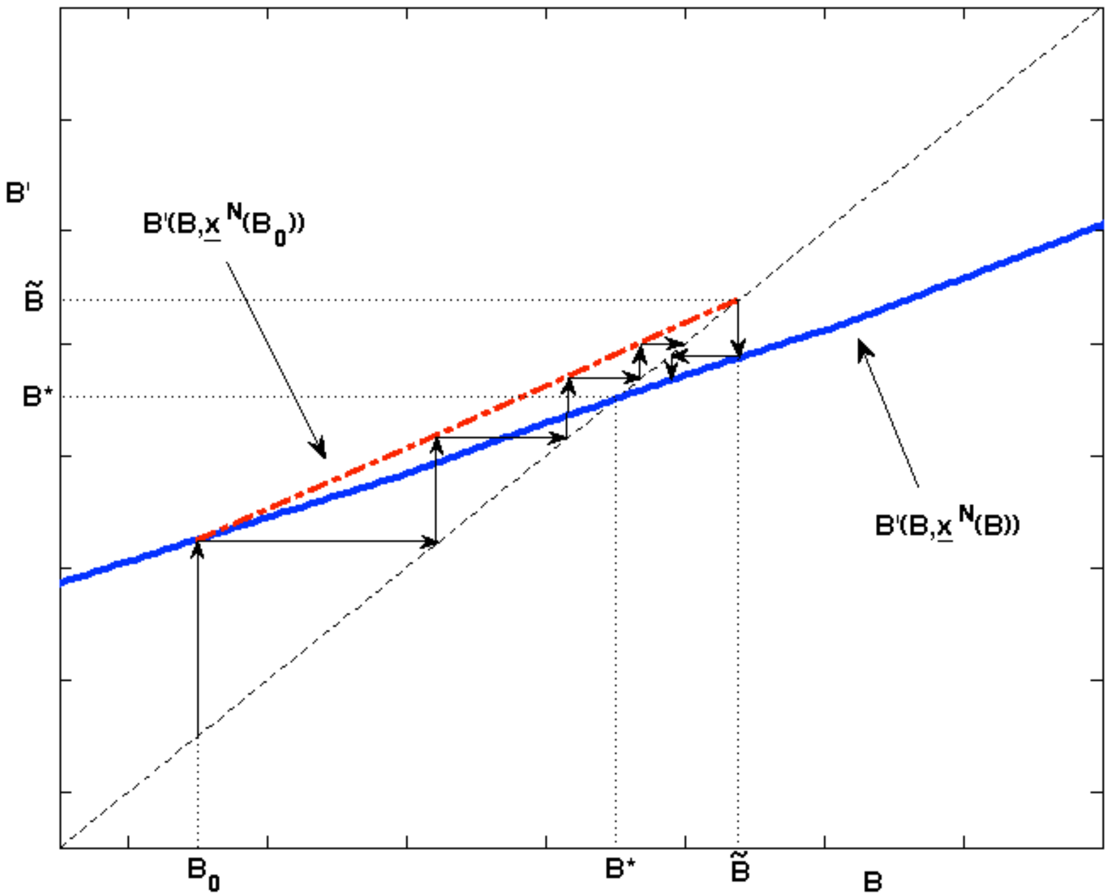
*Proof.* From Lemma 4 we know that $B'(s^j, B, x) = B'(B, x')$. If $j > k$, and the optimal intervals for these two states do not overlap given $B$, then $x'$ must be higher in state $s^j$ than in state $s^k$. Then, the result follows from Lemma 5. If the optimal intervals overlap given $B$, then there exists $x$ for which $x' = x$ in both states $s^j$ and $s^k$. It is clear that aggregate savings are identical in the two states in this case. □

Note that Proposition 6 is relevant both in the long run in the case where assets remain stochastic, and in the short run in all cases before assets converge to $B^*$.

Figure 3 illustrates the short-run dynamics of assets in the case where assets are constant in the long run. The solid (blue) line represents $B'\left(B, \underline{x}^N(B)\right)$, i.e. we compute $B'$ assuming that the relevant participation constraint is binding. Suppose that state $N$ occurs when inherited assets are at the level $B_0 < B^*$. Then public storage will be $B'\left(B_0, \underline{x}^N(B_0)\right)$. Next period, if any state $s^j$ with $j \geq 2$ occurs, no participation constraint is binding, thus assets will be $B'\left(B, \underline{x}^N(B_0)\right) > B'\left(B, \underline{x}^N(B)\right)$. This is represented by the dot-dashed (red) line. As long as state $s^1$ does not occur, assets stay on this line and eventually converge to the level $\widetilde{B} > B^*$. Now, assume that state $s^1$ occurs when inherited assets are $\widetilde{B}$. By symmetry, storage will be $B'\left(\widetilde{B}, \underline{x}^N\left(\widetilde{B}\right)\right)$. Next period, if either state $s^1$ or $s^N$ occurs, the participation constraint binds again, and assets follow the solid (blue) line. If any other state occurs, assets will decrease more, and if this continues happening, assets will approach a level lower than $B^*$ (not represented).

We now characterize the bounds of the stationary distribution of assets when they are stochastic in the long run. Let $\underline{B}$ $\left(\overline{B}\right)$ denote the lower (upper) limit of the stationary distribution of assets.

Figure 3: Short-run asset dynamics when assets are constant in the long run




**Proposition 7.** *The lower limit of the stationary distribution of aggregate assets, $\underline{B}$, is either strictly positive and is implicitly given by*

$$u'\left(\overline{c}^{m}\left(\underline{B}\right)\right)=\beta(1+r)\sum_{j=1}^{N}\pi^{j}\frac{u'\left(c^{j}\left(\underline{B},\overline{x}^{m}\left(\underline{B}\right)\right)\right)}{1-\upsilon^{j}\left(\underline{B},\overline{x}^{m}\left(\underline{B}\right)\right)}, \tag{34}$$

*where the upper index $m$ refers to the least unequal income state, or is zero, and the Euler constraint above holds as strict inequality. The upper limit of the stationary distribution of aggregate assets, $\overline{B}$, is implicitly given by*

$$u'\left(c^{N}\left(\overline{B},\underline{x}^{N}\left(\underline{B}\right)\right)\right)=\beta(1+r)\sum_{j=1}^{N}\pi^{j}u'\left(c^{j}\left(\overline{B},\underline{x}^{N}\left(\underline{B}\right)\right)\right). \tag{35}$$

*Proof.* From Proposition 6 it is clear that $\underline{B}$ will be approached if the least unequal income state, denoted $s^m$,[12] happens repeatedly, while $\overline{B}$ can only be approached with state $s^N$ (or

[12]Note that $s^m$ refers to two states when $N$ is even, $s^{N/2}$ and $s^{N/2+1}$.

$s^1$) happening many times in a row.

If $B$ is part of the stationary distribution, then it must be that $B \geq \underline{B}$. This means that there are less and less resources available over time while assets approach $\underline{B}$, thus the relevant participation constraint will always bind along this path. Therefore, $x' = \overline{x}^m(B)$ along this path, and the planner's Euler is (34) if $\underline{B} > 0$, or $\underline{B} = 0$.[13]

The upper limit of the stationary distribution, $\overline{B}$, is approached from below, thus along that path, the relevant participation constraint is slack.[14] As a result, when $B$ converges to its upper limit, $\tilde{x} \equiv x' = \underline{x}^h(B_1)$ where $B_1$ is the level of inherited assets when we switched to state $s^N$ (or $s^1$). Denote by $\widetilde{B}$ a level of assets where $B$ might converge to from below when state $s^N$ occurs many times in a row. $\widetilde{B}$ is the solution to the following system:

$$\frac{u'\left(c^1\left(\widetilde{B},\tilde{x}\right)\right)}{u'\left(c^N\left(\widetilde{B},\tilde{x}\right)\right)} = \tilde{x}$$

$$c^N\left(\widetilde{B},\tilde{x}\right) + c^1\left(\widetilde{B},\tilde{x}\right) = \bar{Y} + r\widetilde{B}$$

$$u'\left(c^N\left(\widetilde{B},\tilde{x}\right)\right) = \beta(1+r)\sum_{j=1}^{N}\pi^j u'\left(c^j\left(\widetilde{B},\tilde{x}\right)\right). \tag{36}$$

When is $\widetilde{B}$ equal to $\overline{B}$, the upper limit of the stationary distribution? Using Lemma 5, we know that $B'(B,\tilde{x})$ is highest when $\tilde{x}$ is highest. At which asset level $B_1$ within the stationary distribution of assets should we switch to state $s^N$ in order to have $\tilde{x} = \underline{x}^N(B_1)$ the highest possible? This happens when $B_1$ is at the lower limit of the stationary distribution, i.e. when $B_1 = \underline{B}$. In that case, $\tilde{x} = \underline{x}^N\left(\underline{B}\right)$. Then, replacing for $\tilde{x}$ in (36) gives (35). □

Figure 4 illustrates the dynamics of aggregate assets in the case where they are stochastic in the long run. For simplicity, we consider three income states. This means that there are two types of states: two with high income and consumption inequality (states $s^h$ and $s^l$) and one with low income and consumption inequality (state $s^m$). The solid (red) line represents $B'\left(B, \underline{x}^h(B)\right)$, i.e. storage in state $s^h$ (or $s^l$) when the relevant participation constraint is binding. Similarly, the dot-dashed (blue) line represents $B'\left(B, \overline{x}^m(B)\right)$, i.e. storage in state $s^m$ when the relevant participation constraint is binding. Starting from $B_0$, if state $s^m$ occurs repeatedly, assets converge to the lower limit of the stationary distribution, $\underline{B}$. The relevant participation constraint is always binding along this path, because inherited assets

[13]We will give an example for each of these cases in the next section, where we present some computed examples.

[14]It will become clear that the upper limit is reached if state $s^N$ or state $s^1$ occurs many times in a row, but not if the economy alternates between these two states.

keep decreasing. The dashed (green) line represents the scenario when state $s^h$ (or state $s^l$) occurs when inherited assets are at the lower limit of the stationary distribution, $\underline{B}$, and then the same state occurs repeatedly. This is when assets will approach the upper limit of the stationary distribution, $\overline{B}$. The relevant participation constraint is not binding from the period after the switch to $s^h$, therefore storage given inherited assets is described by the function $B'\left(B, \underline{x}^h(\underline{B})\right)$. Finally, without loss of generality assume that state $s^l$ occurred many times while approaching $\overline{B}$, and suppose that state $s^h$ occurs when inherited assets are (close to) $\overline{B}$. In this case, $x' = \underline{x}^h\left(\overline{B}\right) < \underline{x}^h\left(\underline{B}\right)$, and assets will decrease. They will then converge to a level $\underline{\widetilde{B}}$ from above with the relevant participation constraint binding along this path. The same will happen whenever $B > \underline{\widetilde{B}}$ when we switch to state $s^h$ (or $s^l$). $\underline{\widetilde{B}}$ is implicitly given by

$$u'\left(\underline{c}^h\left(\underline{\widetilde{B}}\right)\right) = \beta(1+r)\sum_{j=1}^{N}\pi^j u'\left(c^j\left(\underline{\widetilde{B}}, \underline{x}^h\left(\underline{\widetilde{B}}\right)\right)\right).$$

### 4.1.2 The dynamics of individual consumptions

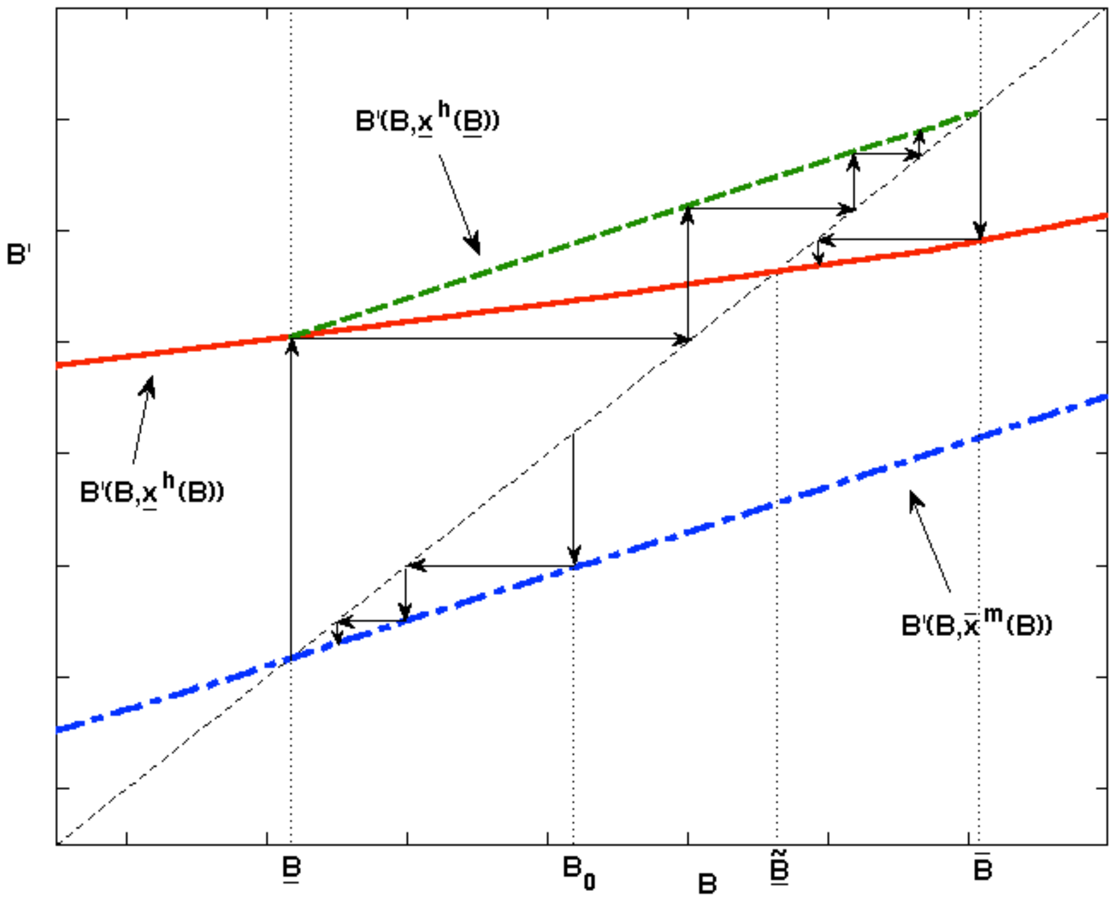
Having characterized assets, we now turn to the dynamics of consumption. One key property of the basic model is that whenever either agent's participation constraint binds ($\upsilon_1(X) > 0$ or $\upsilon_2(X) > 0$), the resulting allocation is independent of the preceding history. In our formulation, this implies that $x'$ is only a function of $s^j$ and the identity of the agent with a binding participation constraint. This is often called the amnesia property (Kocherlakota, 1996), and typically data do not support this pattern, see Broer (2011) for the United States and Kinnan (2011) for Thai villages. Allowing for storage helps to bring the model closer to the data in this respect.

**Proposition 8.** *The amnesia property does not hold when aggregate assets are stochastic in the long run.*

*Proof.* $x'$ and hence current consumption depend on both current income and inherited assets, $B$, when a participation constraint binds. This implies that the past history of income realizations affects current consumption through $B$. □

Another property of the basic model is that whenever neither participation constraint binds ($\upsilon_1(X) = \upsilon_2(X) = 0$), the consumption allocation is constant and hence exhibits an extreme form of persistence. This can be seen easily: (27) gives $x' = x$, and the consumption allocation is only a function of $x'$ with constant aggregate income. This implies that for

Figure 4: Asset dynamics when assets are stochastic in the long run




'small' income changes which do not trigger a participation constraint to bind, we do not see any change in individual consumptions. It is again not easy to find evidence for this pattern in the data. In our model, even if the relative Pareto weight does not change, (21) does not imply that individual consumptions will be the same tomorrow as today. This is because $(1+r)B - B'(X)$ is generically not equal to $(1+r)B' - B''(X')$ when assets are stochastic in the long run.

**Proposition 9.** *The persistence property does not hold when aggregate assets are stochastic in the long run.*

*Proof.* Even though $x' = x$, when neither participation constraint binds, consumption is only constant if net savings are identical in the past and the current period. This is not the case when $B$ is stochastic. □

The last two propositions imply that the dynamics of consumption in the our model are richer and closer to the data than in the basic model. It is also important to note that, for developed economies, the Euler constraint cannot be rejected in micro data, at least in its inequality form, once labor supply decisions and demographics are appropriately accounted for (see Attanasio (1999) for a comprehensive review of the literature). Since in our model agents' Euler inequality is satisfied by construction, we bring limited commitment models in line with this observation as well.

## 4.2 Validity of the first-order condition approach

Until now we have assumed that by introducing the agents' participation and Euler constraints (equations (16) and (17), respectively) in Problem *P1*, we guarantee incentive compatibility. In other words, we have assumed that the constrained-optimal allocation can be obtained by checking that agents have no incentive to default given that they do not save, and that they have no incentive to save given that they do not default. In principle, they may still find it optimal to use more complicated 'double' deviations involving both storage and default, potentially in different time periods, given some history of income shocks.

First, we need to consider contemporaneous joint deviations: the agent defaults and saves at the same time.[15] Since in the participation constraint (16) we use $\tilde{U}_i^{au}(s_t)$, the value of autarky when the agent can save (see equation (19)), this contemporaneous double deviation is already taken into account. Further, note that in autarky the agent is allowed to save whenever this makes him better off. Therefore, the 'default today and save later'-type of double deviations are already taken care of as well. This implies that the only potentially profitable double deviations we still need to consider are those which involve private asset accumulation first and default in a later period.

We demonstrate that such deviations cannot be profitable in the simplest possible case: only two consumption levels occur in the long run, $c^h$ and $c^l$ with $c^h > c^l$ and switching probability $\pi^e$. It is not difficult to generalize the argument for more consumption levels. Let $V^h$ denote the expected lifetime utility of an agent who consumes $c^h$ today. Since $c^h$ is pinned down by the binding participation constraint of agents when their income reaches its highest level, we know that $V^h = V_1^{au}\left(s^N, 0\right)$, where $V_i^{au}()$ is defined in equation (19).

Now, we formally define the problem of an agent who is facing this consumption process and has the option of storing today and defaulting later. We denote the value function

[15] In the literature with private information, a similar joint deviation, shirking (or reporting a lower income) and saving, is the relevant deviation. Detailed discussion of these joint deviations can be found for the hidden action (dynamic moral hazard) case in Kocherlakota (2004) and Ábrahám, Koehne, and Pavoni (2011), and for the hidden income case in Cole and Kocherlakota (2001).

for an agent who is entitled to receive $c^h$ ($c^l$) in the current period, has $b$ units of assets accumulated, and decides not to default today by $W^h(b)$ ($W^l(b)$). These value functions are defined recursively as

$$W^h(b) = \max_{b'\geq 0}\left\{u(c^h + (1+r)b - b') + \beta\left[\sum_{j=2}^{N}\pi^j \max\{W^h(b'), V_1^{au}\left(s^j, b'\right)\}\right.\right.$$
$$\left.\left. + \pi^1 \max\{W^l(b'), V_1^{au}\left(s^1, b'\right)\}\right]\right\} \tag{37}$$

and

$$W^l(b) = \max_{b'\geq 0}\left\{u(c^l + (1+r)b - b') + \beta\left[\sum_{j=1}^{N-1}\pi^j \max\{W^l(b'), V_1^{au}\left(s^j, b'\right)\}\right.\right.$$
$$\left.\left. + \pi^N \max\{W^h(b'), V_1^{au}\left(s^N, b'\right)\}\right]\right\}. \tag{38}$$

We define the solution of the above optimization problems as $g^h(b)$ and $g^l(b)$, respectively.

**Lemma 6.** $g^h(0) = 0$. *That is, the agent assigned to consume $c^h$ today will not save, even if defaulting later is an option.*

*Proof.* Assume indirectly that $g^h(0) > 0$, that is, the agent saves today but does not default. Two cases are possible: either (i) the agent defaults in some state(s) tomorrow, or (ii) the agent does not default in any state tomorrow but he does later.

In case (i), the agent must default when his income is the highest possible tomorrow, i.e. when he earns $y^N$. Let $c^{au}(y^j, b)$ denote the consumption level chosen by the agent in autarky given that his income is $y^j$ and he has accumulated savings $b$. Remember that $\pi^e = \pi^N = \pi^1$. Storing $g^h(0)$ today and defaulting tomorrow if his income is $y^N$, the agent's Euler inequality is

$$\begin{aligned} u'\left(c^h - g^h(0)\right) = \beta(1+r)&\left[\pi^e u'\left(c^{au}\left(y^N, g^h(0)\right)\right)\right. \\ &+ (1-2\pi^e)\, u'\left(c^h + (1+r)g^h(0) - g^h\left(g^h(0)\right)\right) \\ &\left.+\pi^e u'\left(c^l + (1+r)g^h(0) - g^l\left(g^h(0)\right)\right)\right], \end{aligned} \tag{39}$$

We will show that the agent would want to 'borrow' given this consumption path, which he can do by reducing $g^h(0)$. Given that

$$u'\left(c^h\right) = \beta(1+r)\left[\left(1-\pi^e\right)u'\left(c^h\right) + \pi^e u'\left(c^l\right)\right],$$

a sufficient condition for this is that $c^{au}\left(y^N, g^h(0)\right) > c^h$, $c^h + (1+r)g^h(0) - g^h\left(g^h(0)\right) > c^h$, and $c^l + (1+r)g^h(0) - g^l\left(g^h(0)\right) > c^l$. Consumption cannot decrease in the agent's 'income,' i.e.

it cannot be that he chooses a consumption lower than $c^j$ when he has access to $c^j+(1+r)g^h(0)$ units of the consumption good rather than only $c^j$ units. To see the first condition, we first show that $c^{au}\left(y^N,0\right) > c^h$. Assume indirectly that this is not true. Given that the participation constraint holds with equality when the agent's income is $y^N$, this implies that the benefits of being in the risk sharing arrangement occur today while its costs occur in the future relative to autarky. This in turn implies that risk sharing must increase when the discount factor decreases. This contradicts the folk theorem. Recall that in Proposition 5 we have shown that as $\beta$ increases to a level such that $\beta(1+r)=1$, perfect risk sharing is the long-run outcome. Intuitively, a higher $\beta$ means a better enforcement technology in models of risk sharing with limited commitment. Now, clearly, $c^{au}()$ is increasing in its second argument, therefore we also know that $c^{au}\left(y^N, g^h(0)\right) > c^h$ holds for any $g^h(0)\geq 0$. This means that (39) is a strict inequality, thus the agent wishes to increase current consumption, which he can do by reducing $g^h(0)$. A similar argument can be used if the agent would want to default in more states tomorrow.

In case (ii), substituting in the future Euler equations, we can use an almost identical argument as above. For example, take the case when the agent would save in period 0 and 1 and default in the high state in period 2 only if the income delivered by the optimal allocation ($c^h$) remains high in both periods. The Euler equation in period 0 is

$$\begin{aligned} u'\left(c^h-g^h(0)\right) = \beta(1+r)\left[(1-\pi^e)\,u'\left(c^h+(1+r)g^h(0)-g^h\left(g^h(0)\right)\right)\right. \\ \left. +\pi^e u'\left(c^l+(1+r)g^h(0)-g^l\left(g^h(0)\right)\right)\right]. \end{aligned} \tag{40}$$

When the current state is $h$, the Euler equation in period 1 is

$$\begin{aligned} u'\left(c^h+(1+r)g^h(0)-g^h\left(g^h(0)\right)\right) = \beta(1+r)\left[\pi^e u'\left(c^{au}\left(y^N, g^h\left(g^h(0)\right)\right)\right)\right. \\ +(1-2\pi^e)\,u'\left(c^h+(1+r)g^h\left(g^h(0)\right)-g^h\left(g^h\left(g^h(0)\right)\right)\right) \\ \left. +\pi^e u'\left(c^l+(1+r)g^h\left(g^h(0)\right)-g^l\left(g^h\left(g^h(0)\right)\right)\right)\right], \end{aligned} \tag{41}$$

and when the current state is $l$, it is

$$\begin{aligned} u'\left(c^l+(1+r)g^h(0)-g^l\left(g^h(0)\right)\right) = \beta(1+r)\left[\pi^e u'\left(c^h+(1+r)g^l\left(g^h(0)\right)-g^h\left(g^l\left(g^h(0)\right)\right)\right)\right. \\ \left. +(1-\pi^e)\,u'\left(c^l+(1+r)g^l\left(g^h(0)\right)-g^l\left(g^l\left(g^h(0)\right)\right)\right)\right]. \end{aligned} \tag{42}$$

Using equations (41) and (42) to substitute for the marginal utilities on the right hand side of (40) gives the two-period ahead Euler equation in this case. Note that when the agent neither saves nor defaults for two periods, the two-period ahead Euler equation is given by

$$\begin{aligned} u'\left(c^h\right) &= \beta(1+r)\left[(1-\pi^e)\,\beta(1+r)\left((1-\pi^e)\,u'\left(c^h\right)+\pi^e u'\left(c^l\right)\right)\right. \\ &\quad \left. +\pi^e\beta(1+r)\left((1-\pi^e)\,u'\left(c^l\right)+\pi^e u'\left(c^h\right)\right)\right]. \end{aligned} \tag{43}$$

Now, comparing the right hand sides of (40) after substitution and (43) term by term we can use practically the same argument as above to show that $g^h(0)=0$. □

**Proposition 10.** *The first-order condition approach is valid.*

*Proof.* The first-order condition approach is valid if $V^h = W^h(0)$, $V^l = W^l(0)$, and $g^h(0) = g^l(0) = 0$. It is easy to see that $g^l(0)=0$. Lemma 6 shows that $g^h(0)=0$. Replacing these solutions into (37) and (38), the first two conditions follow. □

## 4.3 Computation

We use the recursive system given by equations (26)-(31) to solve the model numerically. We discretize $x$ and $B$ ($y$ is assumed to take a finite number of values). We have to determine $x'$ and $B'$ on a 3-dimensional grid on $X$. The initial values for $V'(X')$, $c_1(X')$, and $\upsilon_1(X')$ are from the solution of a model where the participation constraints are ignored. We iterate until the value and policy functions converge.

As we proceed, we use the characteristics of the solution. In particular, we know that if agent 1's participation constraint binds at $\tilde{x}$, it will bind at all $x < \tilde{x}$. Similarly, if agent 2's participation constraint binds at $\hat{x}$, it will bind at all $x > \hat{x}$. At each iteration, at each income state and for each $B$, we solve directly for these limits, using (29) and (30) with equality in turn, first assuming that $B' = 0$. Afterwards, we check whether the planner's Euler is satisfied at the limits. If not, we solve a 2-equation system of (29) (or (30)) and (28) with equality, with unknowns $(x', B')$. Finally, we solve for a new $B'$ at points on the $x$ grid where neither participation constraint binds, i.e. at the interior of the optimal interval of the current iteration.

## 4.4 Decentralization

Note that public storage can be thought of as a form of capital, $B$ units of which produce $Y + (1+r)B$ units of output tomorrow and which fully depreciates. Ábrahám and Cárceles-Poveda (2006) show how to decentralize a limited commitment economy with capital accumulation which is very similar to the one studied in this paper. In particular, they introduce competitive intermediaries and show that a decentralization with endogenous debt constraints that are 'not too tight' (that make the agents just indifferent between defaulting and participating), as in Alvarez and Jermann (2000), is possible. In that environment, however, they use a neoclassical production function where wages depend on aggregate capital. This implies that there the value of autarky depends on aggregate capital as well.[16] Ábrahám and Cárceles-Poveda (2006) show that if the intermediaries are subject to endogenously determined capital accumulation constraints, then this externality can be taken into account, and the constrained-efficient allocation can be decentralized as a competitive equilibrium.[17]

In our model, however, public storage does not affect the outside option of the agents. Hence, the results above directly imply that a competitive equilibrium corresponding to the constrained-efficient allocation exists. In particular, households trade in Arrow securities subject to endogenous borrowing constraints that prevent default, and intermediaries also sell these Arrow securities to build up public storage. The key intuition is that equilibrium Arrow security prices take into account binding future participation constraints, as these prices are given by the usual pricing kernel. Moreover, agents will not hold any 'shares' in public storage, hence their autarky value will be unaffected. Finally, no arbitrage or perfect competition will make sure that the intermediaries make zero profits in equilibrium.

# 5 Computed examples

In this section we use the above algorithm to solve for efficient allocations in economies with limited commitment and access to hidden storage. We show that aggregate savings due to the planner's desire to complete markets can be significant in magnitude. We also illustrate the role of the discount factor, $\beta$, and the return on storage, $r$.

To do this, we consider five scenarios, four with $\beta(1+r) < 1$ and one with $\beta(1+r) = 1$ to study the benchmark of an efficient storage technology as well. In each case agents' per-period utility function is of the the CRRA form with a coefficient of relative risk aversion equal to 1, i.e. $u() = \ln()$. Income of both agents is i.i.d. over time, and may take three values, $\{0.2, 0.5, 0.8\}$, with equal probabilities. Remember that income is perfectly negatively correlated across the two agents, thus aggregate income is 1 in all three states. When $\beta(1+r) < 1$, the discount factor may take two values, 0.8 and 0.7, and the interest rate may take two values as well, 0.16 and 0.11. In addition, we consider the case $\beta = 0.8$ and $r = 0.25$.

For the basic model, we determine agents' value of autarky allowing them to save. This

[16]This is also the case in the two-country production economy model of Kehoe and Perri (2004).

[17]Chien and Lee (2010) achieves the same objective by taxing capital instead of using a capital accumulation constraint.

means that agents' outside option is the same in the basic model and our model, thereby we can focus on the effect of storage in equilibrium. In all five scenarios, the solution of the basic model is characterized by partial insurance. In one scenario ($\beta = 0.8$, $r = 0.11$), consumption takes only two values in the long run, while it takes four values in the other four scenarios. Table 1 shows the stationary distribution of consumption for the basic model. Further, in all five cases, agents would want to save at the solution of the basic model. That is, hidden storage matters. Therefore, the planner will save in our model.

Table 1: Stationary distribution of consumption in the basic model with storage in autarky, percent of aggregate income

| $\beta$ | 0.8 | 0.8 | 0.8 | 0.7 | 0.7 |
|---|---|---|---|---|---|
| $r$ | 0.25 | 0.16 | 0.11 | 0.16 | 0.11 |
| $\underline{c}^h$ | 80.00 | 80.00 | 59.51 | 80.00 | 64.84 |
| $\overline{c}^m$ | 54.30 | 51.85 | 59.51 | 50.07 | 57.96 |
| $\underline{c}^m$ | 45.70 | 48.15 | 40.49 | 49.93 | 42.04 |
| $\overline{c}^l$ | 20.00 | 20.00 | 40.49 | 20.00 | 35.16 |

In all cases $u() = \ln()$ and income may take three values, $\{0.2, 0.5, 0.8\}$, with equal probabilities.

Now we turn to the results from our model with storage. First, let us look at the behavior of assets in the long run. Table 2 shows the limits of the stationary distribution of assets. As noted above, assets reach their minimum if state $s^m$ occurs repeatedly, while they reach their maximum if state $s^h$ (or state $s^l$) occurs many times in a row, and given that when we switch to either of the unequal states assets were at their minimum.

Table 2: Stationary distribution of assets, percent of aggregate income

| $\beta$ | 0.8 | 0.8 | 0.8 | 0.7 | 0.7 |
|---|---|---|---|---|---|
| $r$ | 0.25 | 0.16 | 0.11 | 0.16 | 0.11 |
| $\overline{B}$ | $\max\{B_0, 36.97\}$ | 16.56 | 2.90 | 7.41 | 1.38 |
| $\underline{B}$ | 36.97 | 16.56 | 2.90 | 6.43 | 0.00 |

In all cases $u() = \ln()$ and income may take three values, $\{0.2, 0.5, 0.8\}$, with equal probabilities. $B_0$ is the initial level of aggregate assets.

When the discount factor is high ($\beta = 0.8$), the participation constraints in state $s^m$ do not bind in the long run for any of the interest rates considered here. As a result assets are constant in the long run, as Proposition 5 states. When the storage technology is efficient ($r = 0.25$), assets reach 36.97 percent of aggregate income in the long run, if the initial level of assets $B_0 \leq 36.97$. Else, assets stay constant at their initial level. When $\beta(1 + r) < 1$ but the interest rate is still relatively high ($r = 0.16$), the planner's savings amount to 16.56 percent of aggregate income. Note that the presence of storage relaxes the participation constraints sufficiently so that they do not bind in state $s^m$, unlike in the basic model. When the interest rate is relatively low ($r = 0.11$), savings are 2.90 percent of aggregate income.

When the discount factor is low ($\beta = 0.7$), participation constraints bind in all three states, and assets remain stochastic in the long run. When the interest rate is relatively high ($r = 0.16$), savings by the planner to make sure agents do not want to save and to improve risk sharing in the future vary between 6.43 and 7.41 percent of aggregate income. When the interest rate is relatively low ($r = 0.11$), savings vary between 0 and 1.38 percent. This last example shows that 0 can be part of the stationary distribution of aggregate assets.

In terms of how much risk sharing is achieved, an increase in the interest rate reduces risk sharing in the basic model, since it raises the value of autarky, see Table 1. In contrast, a higher return on storage improves risk sharing in our model, see Table 3. The benefits from aggregate storage outweigh the negative effect of the increase in the value of agents' outside option.

Table 3: Stationary distribution of consumption, percent of aggregate income

| $\beta$ | 0.8 | 0.8 | 0.8 | 0.7 | 0.7 |
|---|---|---|---|---|---|
| $r$ | 0.25 | 0.16 | 0.11 | 0.16 | 0.11 |
| $\underline{c}^h$ | 54.62 | 56.67 | 58.14 | $[62.94, 63.41]$ | $[63.97, 64.61]$ |
| $\overline{c}^m$ | 54.62 | 56.67 | 58.14 | $[59.73, 60.57]$ | $[57.81, 59.24]$ |
| $\underline{c}^m$ | 54.62 | 45.97 | 42.18 | $[41.20, 41.30]$ | $[41.91, 42.19]$ |
| $\overline{c}^l$ | 54.62 | 45.97 | 42.18 | $[37.50, 38.48]$ | $[35.19, 36.18]$ |

In all cases $u() = \ln()$ and the income may take three values, $\{0.2, 0.5, 0.8\}$, with equal probabilities.

The role of the discount factor is the same in our model with storage as in the basic model. In particular, a higher $\beta$ implies more risk sharing. In the benchmark case where $\beta = 0.8$ and $r = 0.25$, i.e. when the storage technology is efficient, perfect risk sharing is self-enforcing in the long run. This is because in this case there is no trade-off between the two inefficiencies, namely, imperfect risk sharing and an inferior intertemporal technology.

When the discount factor is low ($\beta = 0.7$), consumption depends on inherited assets, which in turn depend on the history of income shocks. As a result, the amnesia and persistent properties do not hold in the long run in these examples. The variation of consumption is small with a maximum of 2.8 percent in state $s^l$ when $\beta = 0.7$ and $r = 0.11$.

# 6 Concluding remarks

This paper has shown that some implications of the basic limited commitment model with no private or public storage are not robust to hidden storage. When public storage is allowed though, the incentive for private storage is eliminated in the constrained-optimal allocation. The intertemporal technology is used in equilibrium even though there is no aggregate uncertainty and the return is lower than the discount rate, i.e. $\beta(1+r) < 1$. The planner saves for two reasons. She saves to eliminate agents' incentives to use their hidden storage technology. When income inequality is not the highest, the planner has more incentive to save than the agents. The reason for additional storage by the planner is that public assets relax future participation constraints, and thus improve risk sharing.

We have also shown that aggregate assets may be stochastic in the long run. This happens when each agent's participation constraint binds at more than one income level, i.e. when cross-sectional consumption inequality varies over time. Given inherited assets, public storage is higher when consumption inequality is higher. Finally, the dynamics of consumption is richer in our model compared to the basic model without storage. In particular, the amnesia and persistence properties do not hold in general, which brings limited commitment models closer to the data (Broer, 2011). Further, in our model agents' Euler constraints in inequality form hold by construction, which is consistent with empirical evidence from developed countries (Attanasio, 1999).

The literature on incomplete markets either exogenously restrict asset trade, most prominently by allowing only a risk-free bond to be traded (Huggett, 1993; Aiyagari, 1994), or considers a deep friction that limits risk sharing endogenously. The literature has focused on two such frictions, namely limited commitment and private information. This paper merges these two strands of the incomplete markets literature by allowing for state-contingent trading subject to a deep friction and a self-insurance opportunity at the same time. This has been done when the deep friction is hidden income or effort (Allen, 1985; Cole and Kocherlakota, 2001; Ábrahám, Koehne, and Pavoni, 2011), but not in the case where the deep friction is lack of commitment, to our knowledge.

Comparing our model with limited commitment and hidden storage to models with hidden income or effort and hidden storage points to remarkable differences. In our model, if the same storage technology is available to the planner and the agents, welfare is improved.[18] In contrast, in models with hidden income/effort, public storage is never used and the presence of the hidden storage opportunity reduces welfare. This is because in our model saving by the planner both improves insurance and relaxes the incentive problem, by relaxing future participation constraints; while in the hidden income/effort context the incentive problem is made more severe by additional self-insurance, and aggregate asset accumulation makes incentive provision more expensive.

Our model could be applied in several economic contexts. The model predicts that risk sharing among households in villages can be improved by a public grain storage facility. Cooperation among partners in a law firm, for example, is facilitated by common assets that someone quitting the partnership has no access to. Marriage contracts may specify that some commonly held assets are lost by the spouse who files for divorce. Finally, supranational organizations may help international risk sharing by simply having a jointly held stock of assets. The European Financial Stability Facility and the European Stability Mechanism may serve this purpose. Future work should study the quantitative implications of saving using some of these applications.

[18] The planner and the agents can always choose zero assets. Given that in the constrained-optimal allocation the planner has strictly positive assets (at least in some periods), welfare strictly improves.

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